Weak gravitational wave turbulence in the early Universe:

theory & simulation

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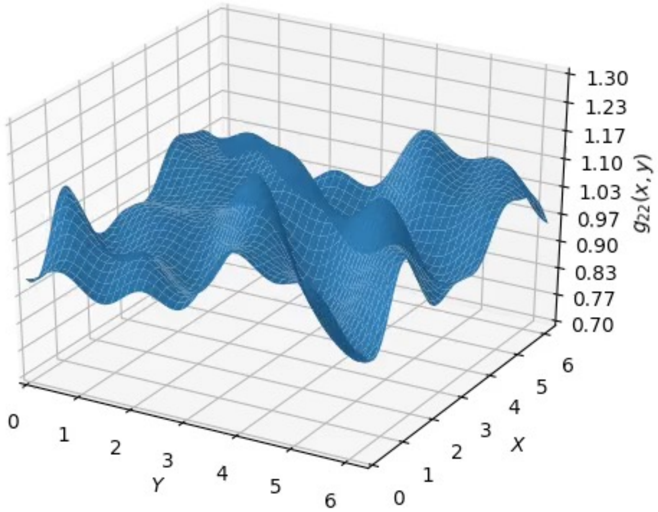


WAVE COMPLEXITY

institut universitaire de France

Selected for a Viewpoint in *Physics*

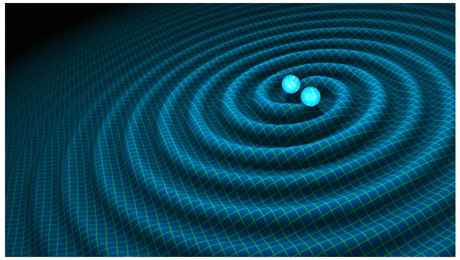
PRL **116,** 061102 (2016) PHYSICAL REVIEW LETTERS week ending 12 FEBRUARY 2016

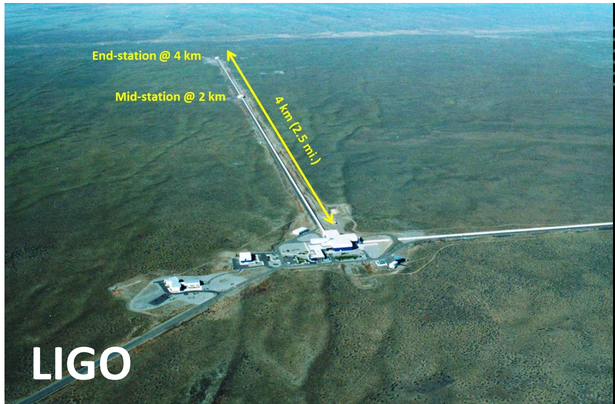
Observation of Gravitational Waves from a Binary Black Hole Merger

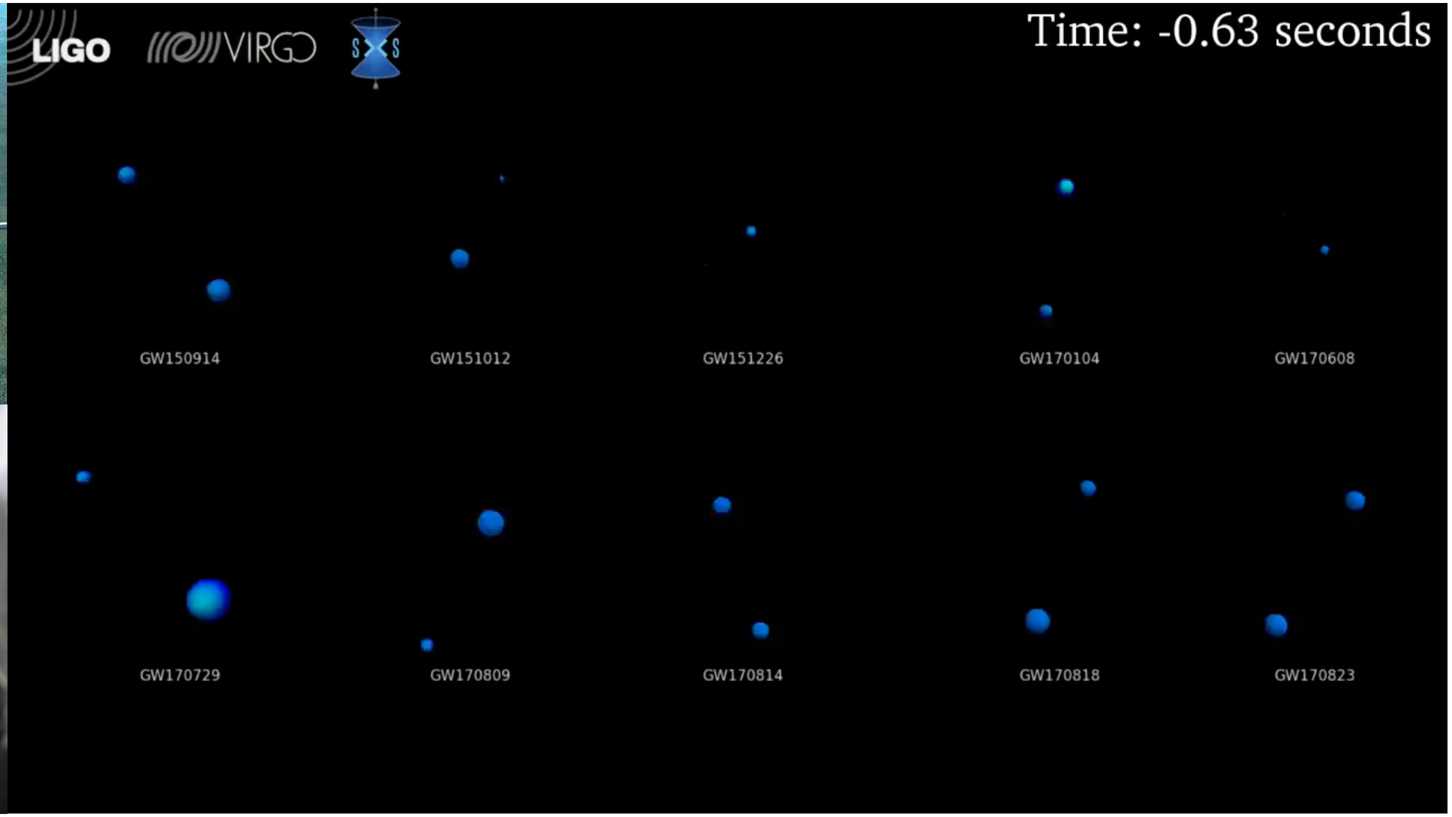
B. P. Abbott *et al.**

(LIGO Scientific Collaboration and Virgo Collaboration)

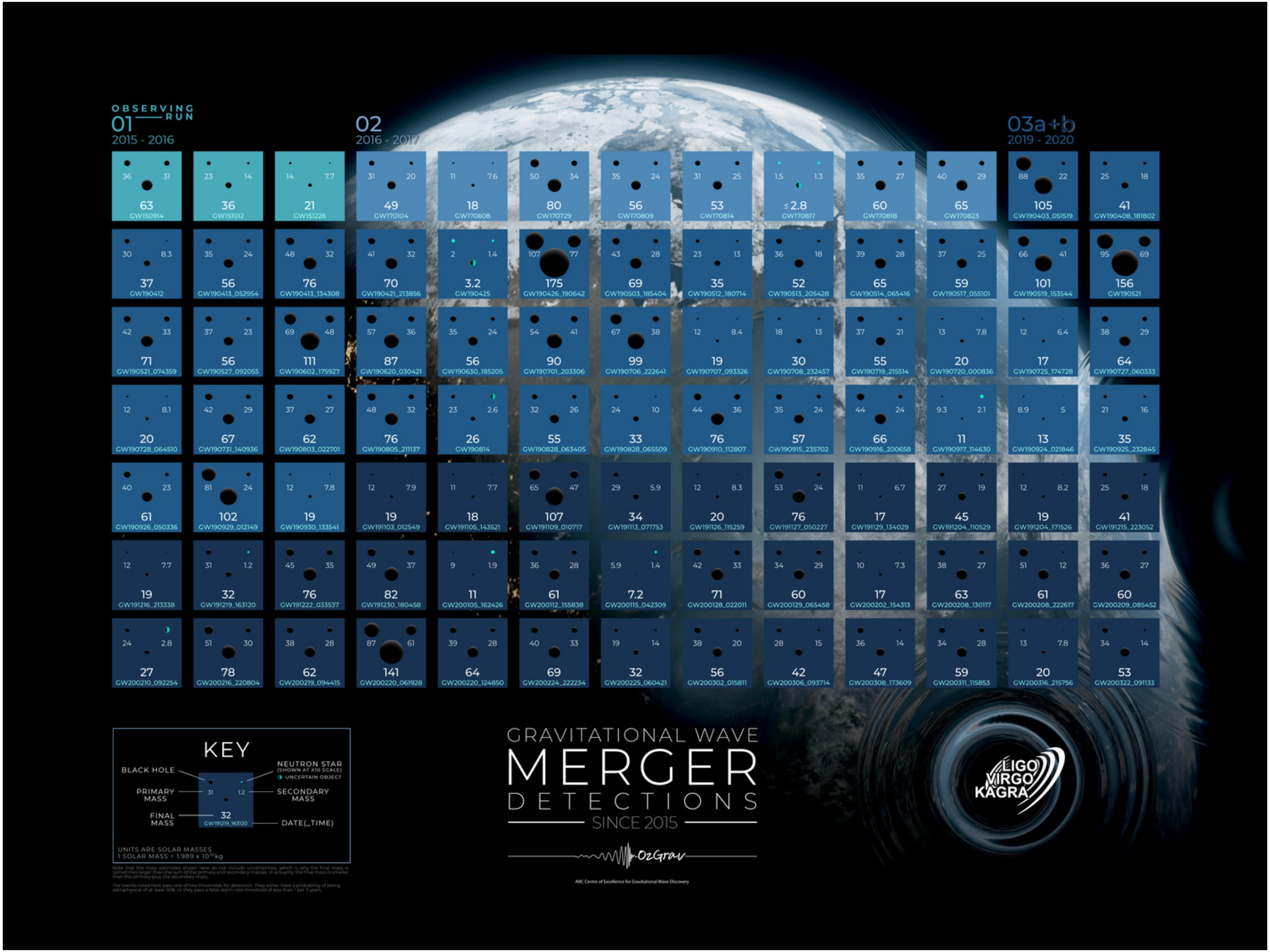
(Received 21 January 2016; published 11 February 2016)



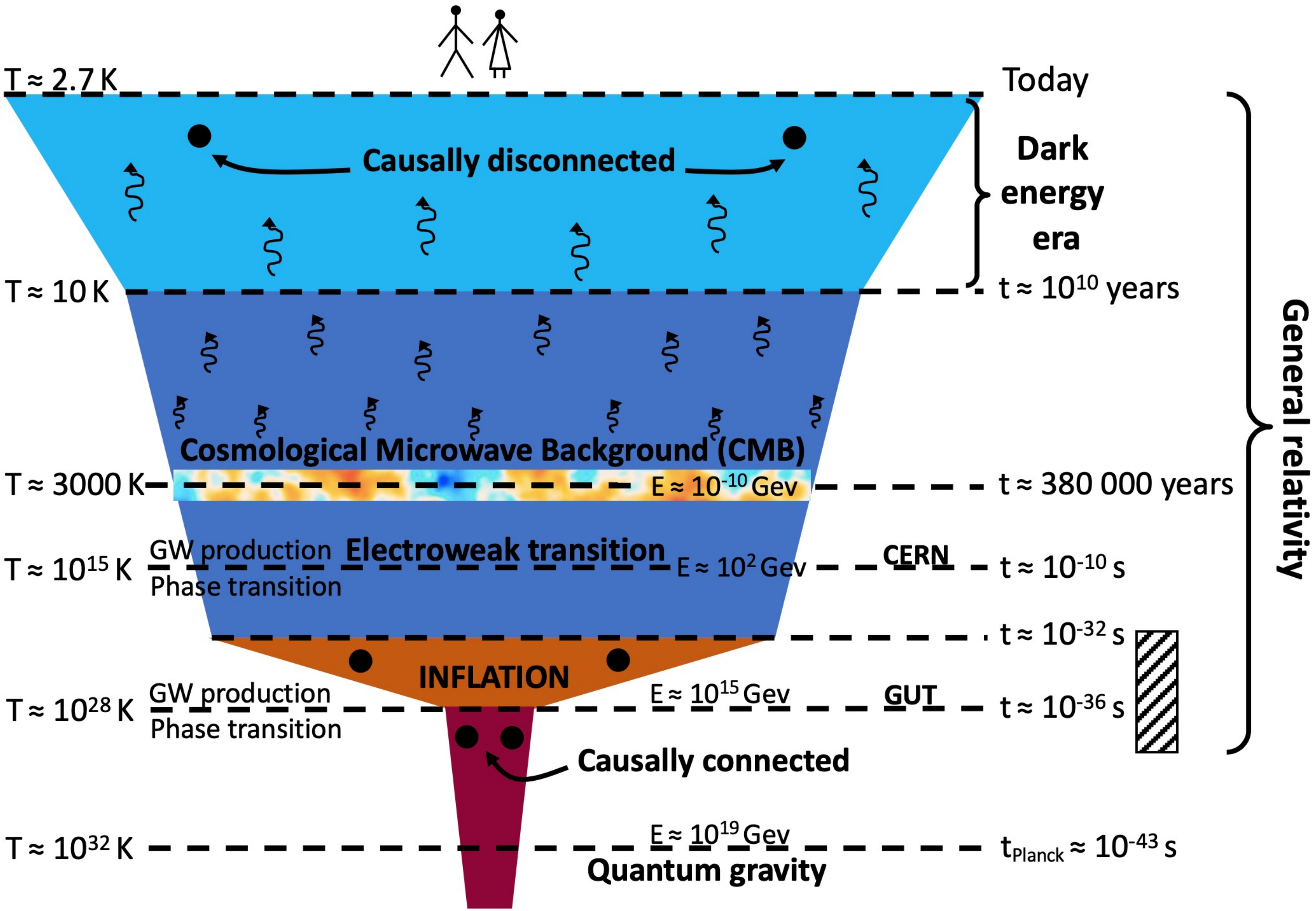




A new window on the universe


OBSERVING RUN
01
2015 - 2016
02
2016 - 2017
O3a+b
2019 - 2020
36 31 63 GW150914
23 14 36 GW151012
14 7.7 21 GW151226
31 20 49 GW170104
11 7.6 18 GW170608
50 34 80 GW170729
35 24 56 GW170809
31 25 53 GW170814
1.5 1.3 ≤2.8 GW170817
35 27 60 GW170818
40 29 65 GW170823
88 22 105 GW190403_051519
25 18 41 GW190408_181802
30 8.3 37 GW190412
35 24 56 GW190413_052954
48 32 76 GW190413_134308
41 32 70 GW190421_213856
2 1.4 3.2 GW190425
107 77 175 GW190426_190642
43 28 69 GW190503_185404
23 13 35 GW190512_180714
36 18 52 GW190513_205428
39 28 65 GW190514_065416
37 25 59 GW190517_055101
66 41 101 GW190519_153544
95 69 156 GW190521
42 33 71 GW190521_074359
37 23 56 GW190527_092055
69 48 111 GW190602_175927
57 36 87 GW190620_030421
35 24 56 GW190630_185205
54 41 90 GW190701_203306
67 38 99 GW190706_222641
12 8.4 19 GW190707_093326
18 13 30 GW190708_232457
37 21 55 GW190719_215514
13 7.8 20 GW190720_000836
12 6.4 17 GW190725_174728
38 29 64 GW190727_060333
12 8.1 20 GW190728_064510
42 29 67 GW190731_140936
37 27 62 GW190803_022701
48 32 76 GW190805_211137
23 2.6 26 GW190814
32 26 55 GW190828_063405
24 10 33 GW190828_065509
44 36 76 GW190910_112807
35 24 57 GW190915_235702
44 24 66 GW190916_200658
9.3 2.1 11 GW190917_114630
8.9 5 13 GW190924_021846
21 16 35 GW190925_232845
40 23 61 GW190926_050336
81 24 102 GW190929_012149
12 7.8 19 GW190930_133541
12 7.9 19 GW191103_012549
11 7.7 18 GW191105_143521
65 47 107 GW191109_010717
29 5.9 34 GW191113_071753
12 8.3 20 GW191126_115259
53 24 76 GW191127_050227
11 6.7 17 GW191129_134029
27 19 45 GW191204_110529
12 8.2 19 GW191204_171526
25 18 41 GW191215_223052
12 7.7 19 GW191216_213338
31 1.2 32 GW191219_163120
45 35 76 GW191222_033537
49 37 82 GW191230_180458
9 1.9 11 GW200105_162426
36 28 61 GW200112_155838
5.9 1.4 7.2 GW200115_042309
42 33 71 GW200128_022011
34 29 60 GW200129_065458
10 7.3 17 GW200202_154313
38 27 63 GW200208_130117
51 12 61 GW200208_222617
36 27 60 GW200209_085452
24 2.8 27 GW200210_092254
51 30 78 GW200216_220804
38 28 62 GW200219_094415
87 61 141 GW200220_061928
39 28 64 GW200220_124850
40 33 69 GW200224_222234
19 14 32 GW200225_060421
38 20 56 GW200302_015811
28 15 42 GW200306_093714
36 14 47 GW200308_173609
34 28 59 GW200311_115853
13 7.8 20 GW200316_215756
34 14 53 GW200322_091133
KEY
BLACK HOLE
NEUTRON STAR (SHOWN AT X10 SCALE)
UNCERTAIN OBJECT
PRIMARY MASS
31
1.2
SECONDARY MASS
FINAL MASS
32
GW191219_163120
DATE(_TIME)
UNITS ARE SOLAR MASSES
1 SOLAR MASS = 1.989 x 10^30kg
Note that the mass estimates shown here do not include uncertainties, which is why the final mass is sometimes larger than the sum of the primary and secondary masses. In actuality, the final mass is smaller than the primary plus the secondary mass.
The events listed here pass one of two thresholds for detection. They either have a probability of being astrophysical of at least 50%, or they pass a false alarm rate threshold of less than 1 per 3 years.
GRAVITATIONAL WAVE
MERGER
DETECTIONS
SINCE 2015
OzGrav
ARC Centre of Excellence for Gravitational Wave Discovery
LIGO
VIRGO
KAGRA

History of the observable universe



Inflation is not explained from first principles (hypothetical *inflaton*)

[Ijjas+, PLB, 2013]

Gravitational wave turbulence

Theory

PRL 119, 221101 (2017) PHYSICAL REVIEW LETTERS week ending 1 DECEMBER 2017

Turbulence of Weak Gravitational Waves in the Early Universe

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(Received 17 October 2016; revised manuscript received 13 October 2017; published 28 November 2017)

We study the statistical properties of an ensemble of weak gravitational waves interacting nonlinearly in a flat space-time. We show that the resonant three-wave interactions are absent and develop a theory for four-wave interactions in the reduced case of a $2.5+1$ diagonal metric tensor. In this limit, where only plus-polarized gravitational waves are present, we derive the interaction Hamiltonian and consider the asymptotic regime of weak gravitational wave turbulence. Both direct and inverse cascades are found for the energy and the wave action, respectively, and the corresponding wave spectra are derived. The inverse cascade is characterized by a finite-time propagation of the metric excitations—a process similar to an explosive nonequilibrium Bose–Einstein condensation, which provides an efficient mechanism to ironing out small-scale inhomogeneities. The direct cascade leads to an accumulation of the radiation energy in the system. These processes might be important for understanding the early Universe where a background of weak nonlinear gravitational waves is expected.

DOI: 10.1103/PhysRevLett.119.221101

Introduction.—The recent direct observations of gravitational waves (GWs) by the LIGO-Virgo collaboration [1], a century after their prediction by Einstein [2], is certainly one of the most important events in astronomy, which opens a new window onto the Universe, the so-called GW astronomy. In the modern Universe, shortly after being excited by a source, e.g., a merger of two black holes, GWs become essentially linear and therefore noninteracting during their subsequent propagation. In the very early Universe, different mechanisms have been proposed for the generation of primordial GWs, like e.g., phase transition [3–9], self-ordering scalar fields [10], cosmic strings [11], and cosmic defects [12]. Production of GWs is also expected to have taken place during the cosmological inflation era [13–15], and many efforts are currently made to detect indirectly their existence [16]. The physical origin of the exponential expansion of the early Universe is, however, not clearly explained and still under investigation [17,18]. Formally, it was incorporated into the general relativity equations simply through adding a positive cosmological constant.

The primordial GWs were, presumably, significantly more nonlinear than the GWs in the modern Universe (like the GWs observed recently by LIGO-Virgo) as they had much larger energy packed in a much tighter space [19]. Although not firmly validated, a scenario was suggested in which a first-order phase transition proceeds through the collisions of true-vacuum bubbles creating a potent source of GWs [20–22]. According to this scenario, at the time of the grand-unified-theory (GUT) symmetry breaking ($t_* \sim 10^{-36}$ sec, $T_* \sim 10^{15}$ GeV), the ratio of the energy density in GW ($\rho_{\rm GW}$) to that in radiation ($\rho_{\rm rad}$) after the transition is about 5% [21]. From the expressions given in [21] and using as a time scale t_* (and also $g_* \sim 100$), we find the following estimate for the GW amplitude: $h \sim 0.3$. Supposedly, such waves were covering the Universe quasiuniformly rather than being concentrated locally in space and time near an isolated burst event, and it is likely that their distribution was broad in frequencies and propagation angles. At some stage of expansion of the Universe, the GWs had become rather weak, but still nonlinear enough for having nontrivial mutual interactions. Importance of the nonlinear nature of the GWs was pointed out in the past for explaining, e.g., the memory effect [23] or part of the dark energy [24]. The possibility to get a turbulent energy cascade of the primordial gravitons was also mentioned [25,26], but, to date, no theory has been developed. A turbulence theory seems to be particularly relevant for GWs because they are nonlinear, and their dissipation is negligible. Recent works [27,28] explore some ideas on similar lines: they investigate numerically the turbulent nature of black holes, define a gravitational Reynolds number, and show that the system can display a nonlinear parametric instability with transfers reminiscent of an inverse cascade (see also Refs. [29,30]).

The nonlinear properties of the GWs, especially the primordial GWs mentioned above, call for using the wave turbulence approach considering statistical behavior of random weakly nonlinear waves [31,32]. The energy transfer between such waves occurs mostly within resonant sets of waves, and the resulting energy distribution, far from a thermodynamic equilibrium, is often characterized by exact power law solutions similar to the Kolmogorov spectrum of hydrodynamic turbulence—the so-called Kolmogorov–Zakharov (KZ) spectra [31,32]. The wave turbulence approach has been successfully applied to many diverse

DNS

PHYSICAL REVIEW LETTERS 127, 131101 (2021)

Editors' Suggestion

Direct Evidence of a Dual Cascade in Gravitational Wave Turbulence

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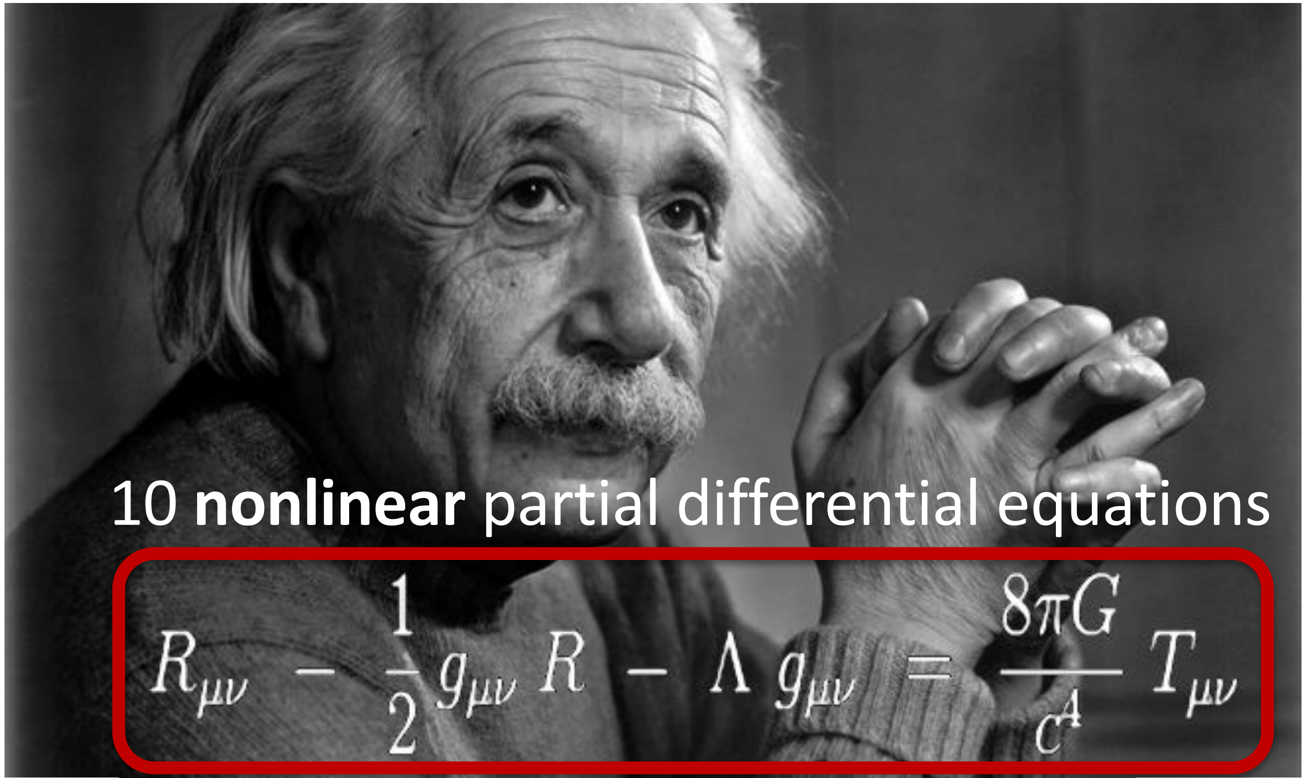
We present the first direct numerical simulation of gravitational wave turbulence. General relativity equations are solved numerically in a periodic box with a diagonal metric tensor depending on two space coordinates only, $g_{ij} \equiv g_{ii}(x, y, t)\delta_{ij}$, and with an additional small-scale dissipative term. We limit ourselves to weak gravitational waves and to a freely decaying turbulence. We find that an initial metric excitation at intermediate wave number leads to a dual cascade of energy and wave action. When the direct energy cascade reaches the dissipative scales, a transition is observed in the temporal evolution of energy from a plateau to a power-law decay, while the inverse cascade front continues to propagate toward low wave numbers. The wave number and frequency-wave-number spectra are found to be compatible with the theory of weak wave turbulence and the characteristic timescale of the dual cascade is that expected for four-wave resonant interactions. The simulation reveals that an initially weak gravitational wave turbulence tends to become strong as the inverse cascade of wave action progresses with a selective amplification of the fluctuations g_{11} and g_{22}.

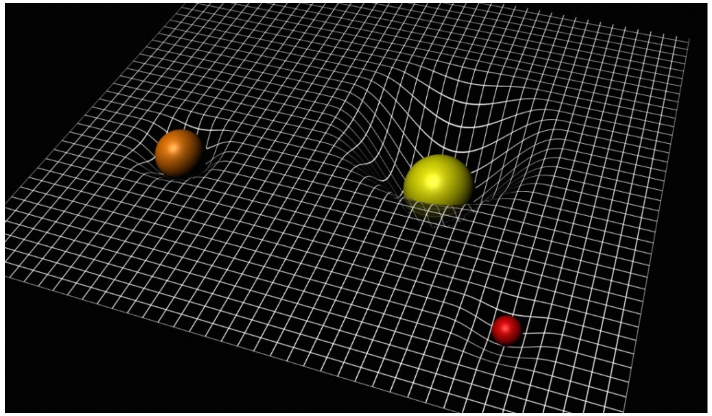
DOI: 10.1103/PhysRevLett.127.131101

Introduction.—Wave turbulence (WT) is a state of a continuous medium with random mutually interacting waves of weak amplitude excited over a broad range of wave numbers. The long-time statistical properties of such a medium have a natural asymptotic closure induced by the large separation of linear and nonlinear timescales [1–3]. The dynamics of WT is driven by kinetic equations which describe the redistribution of spectral densities via mainly three- or four-wave resonant interactions. The kinetic equations have two types of exact stationary power-law solutions: the zero-flux equilibrium thermodynamic spectra and the finite flux nonequilibrium Kolmogorov-Zakharov spectra [4]. The latter solutions are much more interesting because they describe the spectral transfer of conserved quantities, such as energy or wave action, generally between a source and a sink [5,6]. The direction of the cascade, direct or inverse, can be found by a numerical evaluation of the sign of the associated flux. The theory also offers the possibility to predict the Kolmogorov constant. All these properties makes WT a very interesting regime to understand the mechanisms underlying turbulence in depth.

WT is of interest to many physical systems for which theoretical predictions have been made and numerically or experimentally verified. We have, among others, capillary waves [7–14] and gravity waves [15–18] on fluid surfaces, inertial waves in rotating hydrodynamics [19–26], elastic waves on thin vibrating plates [27–32], optical waves in optical fibers [33,34], waves in Bose-Einstein condensate [35,36], Kelvin waves on quantum vortex filaments [37–39], magnetostrophic waves in geodynamo [40,41] and magnetohydrodynamic waves in space plasmas [42–47]. Recently, a theory of WT has been developed for gravitational waves (GWs) [48], a few years after their first direct detection [49]. A promising application concerns the primordial universe shortly after the hypothetical initial singularity. During this period, GWs can be produced by different mechanisms like, e.g., first order phase transition [50,51] or the merger of primary black holes which can be formed from the primordial space-time fluctuations [52]. A typical length scale of GW excitation can be 10^{-30} m. Following this idea, a scenario of cosmological inflation has been proposed relying on the presence of weak or strong GW turbulence and rapid formation of a condensate via an inverse cascade [53]. In this scenario, the initial strong GW bursts are quickly diluted as they propagate through the surrounding space, resulting in a statistically quasihomogeneous GW field that is weakly or strongly nonlinear depending on the strength and density of forcing events. For the weak WT case, a kinetic equation that describes the dynamics of energy and wave action via four-wave resonant interactions was derived. It has exact stationary scaling solutions for the one-dimensional (1D) isotropic spectrum of wave action: k^{-1} for the direct energy cascade and $k^{-2/3}$ for the inverse wave action cascade. Further, an explosive front propagation in the inverse cascade is predicted and numerically observed with a phenomenological nonlinear diffusion model where strongly local interactions are retained [54]. With this model, it is also shown that the nonstationary isotropic spectrum of wave

Einstein's equations





$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$$

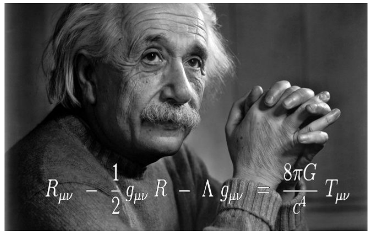
$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left[\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk} \right]$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

[Einstein, 1915]

- $R_{\mu\nu}$: Ricci tensor
- R : Ricci scalar
- $g_{\mu\nu}$: metric tensor
- Λ : cosmological constant

- $\Gamma^\alpha_{\mu\nu}$: Christoffel symbol
- $T_{\mu\nu}$: stress-energy
- $G = 6.67\ 10^{-11}\ m^3 kg^{-1} s^{-2}$
- $c = 2.99\ 10^8\ ms^{-1}$



Gravitational waves (GW)

$\Lambda=0$

Exact linear solutions in an empty – flat – Universe: $\boxed{R_{\mu\nu} = 0}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } |h_{\mu\nu}| \ll 1$$

Poincaré-Minkowski space-time metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Effect of a + gravitational wave on a ring of particles (h = 0.5)

$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

[Einstein, 1916]

$$h^{+}_{\mu\nu} = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Weak wave turbulence theory

Statistical description for weakly non-linear waves

Analytical theory in spectral space

+ Natural **asymptotical closure** of the hierarchy of moment equations
  [Benney & Saffman, PRSLA, 1966; Benney & Newell, JMP, 1967]

+ The kinetic equations admits exact **stationary finite flux** solutions
  [Zakharov & Filonenko, DAN, 1966]

– Finite flux spectra not valid **over all k's** $\rightarrow$ strong wave turbulence
  [Galtier+, JPP, 2000; Meyrand+, PRL, 2016]

– Experiments and DNS show **some limitations** in the predictions
  [Morize+, PoF, 2005; Nazarenko, NJP, 2007]

Weakly nonlinear general relativity

$\Lambda=0$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1 \qquad R_{\mu\nu} = 0$$

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \cancel{R^{(3)}_{\mu\nu} + R^{(4)}_{\mu\nu} + \dots} \qquad R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}$$

<u>Triadic interactions:</u>

$$\begin{cases} \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \text{ and } \omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} \\ \omega_{\mathbf{k}} = c|\mathbf{k}| = ck \end{cases}$$

$\Rightarrow$ **Collinear wave vectors**

We found no contribution on the **resonant manifold**

Three-wave interactions in GW turbulence **does not contribute**!

[SG & Nazarenko, PRL, 2017]

Weakly nonlinear general relativity Λ=0

Einstein-Hilbert action: $S = \frac{1}{2}\int R\sqrt{-g}\underline{d^4x}$ g is the determinant of $g_{\mu\nu}$

R is the scalar curvature

Diagonal space-time metric:

$\partial/\partial z = 0$

$$g_{\mu\nu} = \begin{pmatrix} -(H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{pmatrix}$$

[Hadad & Zakharov, JGP, 2014]

$$H_0 = e^{-\lambda}\gamma, \;\; H_1 = e^{-\lambda}\beta, \;\; H_2 = e^{-\lambda}\alpha, \;\; H_3 = e^{\lambda}$$

Valid for any GW amplitude but we will limit ourselves to weak amplitude

Lagrangian density:

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}\left[\overbrace{\frac{\alpha\beta}{\gamma}\dot{\lambda}^2 - \frac{\alpha\gamma}{\beta}(\partial_x\lambda)^2 - \frac{\beta\gamma}{\alpha}(\partial_y\lambda)^2}^{\text{Give the linear contribution}} - \frac{\dot{\alpha}\dot{\beta}}{\gamma} + \frac{(\partial_x\alpha)(\partial_x\gamma)}{\beta} + \frac{(\partial_y\beta)(\partial_y\gamma)}{\alpha}\right]$$

$$\alpha = \beta = \gamma = 1 \qquad \lambda \ll 1 \qquad \lambda = c_1\exp(-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}) + c_2\exp(i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x})$$

Hadad & Zakharov's theorem (JGP, 2014)

Valid for any gravitational wave amplitude

- Dynamical equations given by: $$\begin{cases} \dfrac{\delta S}{\delta \lambda} = 0 \\ \dfrac{\delta S}{\delta \alpha} = \dfrac{\delta S}{\delta \beta} = \dfrac{\delta S}{\delta \gamma} = 0 \end{cases}$$

4 equations

- Vacuum Einstein equations:

7 equations

$$[R_{\mu\nu}] = \begin{pmatrix} R_{00} & R_{01} & R_{02} & 0 \\ - & R_{11} & R_{12} & 0 \\ - & - & R_{22} & 0 \\ - & - & - & R_{33} \end{pmatrix} = [0]$$

$\partial/\partial z = 0$

It's compatible !

General relativity equations for GW turbulence

$$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \lambda \ll 1$$

$$\boxed{\begin{gathered}\partial_x \dot{\tilde{\alpha}} = -2\dot{\lambda}(\partial_x \lambda), \quad \partial_y \dot{\tilde{\beta}} = -2\dot{\lambda}(\partial_y \lambda) \quad \partial_x \partial_y \tilde{\gamma} = -2(\partial_x \lambda)(\partial_y \lambda) \\ \partial_t[(1+\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})\dot{\lambda}] = \partial_x[(1+\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})\partial_x \lambda] + \partial_y[(1-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})\partial_y \lambda]\end{gathered}}$$

$$g_{00} = -(1+\tilde{\gamma})^2 e^{-2\lambda}$$

$$g_{11} = (1+\tilde{\beta})^2 e^{-2\lambda}$$

$$g_{22} = (1+\tilde{\alpha})^2 e^{-2\lambda}$$

$$g_{33} = e^{2\lambda}$$

Hamiltonian formalism

Normal variables: $\lambda_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a^*_{-\mathbf{k}}}{\sqrt{2k}}, \quad \dot{\lambda}_{\mathbf{k}} = \frac{\sqrt{k}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})}{i\sqrt{2}},$ (Fourier space)

Hamiltonian equation: $i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}}$ where $H = H_{\text{free}} + H_{\text{int}}$

$$H_{\text{free}} = \sum_{\mathbf{k}} k|a_{\mathbf{k}}|^2$$

With $R_{01}=R_{02}=R_{12}=0$ we find:

$$\begin{aligned}
H_{\text{int}} = \frac{1}{4}\sum_{1,2,3,4,5} \frac{\delta_{123}\delta^1_{45}}{\sqrt{k_2k_3k_4k_5}} \Bigg\{ \Bigg[\left(\frac{p_5}{p_1} + \frac{q_5}{q_1}\right) k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5 + k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5 - k_4} \right) + \frac{p_4q_5}{p_1q_1}(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \Bigg] \\
k_2k_3(a_2 - a^*_{-2})(a_3 - a^*_{-3}) + \Bigg[-\left(\frac{p_5}{p_1} - \frac{q_5}{q_1}\right)(p_2p_3 - q_2q_3)k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5 + k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5 - k_4} \right) \\
+ \frac{p_4q_5}{p_1q_1}(\mathbf{k}_2 \cdot \mathbf{k}_3)(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \Bigg] (a_2 + a^*_{-2})(a_3 + a^*_{-3}) \Bigg\} \\
+ \frac{1}{2}\sum_{\mathbf{k},1,2,3,4} \frac{\delta^{\mathbf{k}}_{12}\delta^{\mathbf{k}}_{34}}{\sqrt{k_1k_2k_3k_4}} \Bigg\{ \frac{(\mathbf{k}\cdot\mathbf{k}_2)k_1p_3q_4}{pq} \left(-\frac{a_1a_2 + a^*_{-1}a^*_{-2}}{k_2 + k_1} + \frac{a^*_{-1}a_2 + a_1a^*_{-2}}{k_2 - k_1} \right) (a^*_3 + a_{-3})(a^*_4 + a_{-4}) \\
+ \frac{k_1k_3p_2q_4}{pq} \left(a_1a_2 + a_1a^*_{-2} - a^*_{-1}a_2 - a^*_{-1}a^*_{-2}\right)\left(a^*_3a^*_4 + a^*_3a_{-4} - a_{-3}a^*_4 - a_{-3}a_{-4}\right) \Bigg\}. \qquad (6)
\end{aligned}$$

4 wave processes

Kinetic equation (4-wave interactions)

$$n_{\mathbf{k}} = \langle |a_{\mathbf{k}}|^2 \rangle$$

$$\boxed{H_{3\to 1} = 0}$$ $\rightarrow$ Additional symmetry

$$\boxed{\dot{n}_{\mathbf{k}} = 4\pi \int |T^{\mathbf{k}\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}|^2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right] \delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,}$$

$$\text{with } T^{12}_{34} = \tfrac{1}{4}(W^{12}_{34} + W^{21}_{34} + W^{12}_{43} + W^{21}_{43}),\; W^{12}_{34} = Q^{12}_{34} + Q^{34}_{12}$$

$$Q^{12}_{34} = \frac{1}{4\sqrt{k_1 k_2 k_3 k_4}} \left\{ 2\left(\frac{p_4}{p_1 - p_3} - \frac{q_4}{q_1 - q_3} \right) \frac{k_2(p_1 p_3 - q_1 q_3)}{k_1 - k_3} - 2\left(\frac{p_4}{p_1 - p_3} + \frac{q_4}{q_1 - q_3} \right) \frac{k_1 k_2 k_3}{k_1 - k_3} \right. \tag{12}$$
$$\left. + \left(\frac{p_2}{p_1 + p_2} - \frac{q_2}{q_1 + q_2} \right) \frac{k_1(p_3 p_4 - q_3 q_4)}{k_1 + k_2} - \left(\frac{p_2}{p_1 + p_2} + \frac{q_2}{q_1 + q_2} \right) \frac{k_1 k_3 k_4}{k_1 + k_2} + \frac{2 k_1 k_3 p_2 q_4}{(p_1 + p_2)(q_1 + q_2)} + \frac{2 k_1 p_3 (q_2 k_4 + k_2 q_4)}{(p_1 - p_3)(q_1 - q_3)} \right\}.$$

[SG & Nazarenko, PRL, 2017]

Constant (non-zero) flux (isotropic) spectra:

Energy

$$\mathcal{E} = \iint \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} = \text{const}$$

$$\boxed{E^{(1D)}_k \sim \varepsilon^{1/3} k^0}$$

Direct cascade

Wave action

$$\mathcal{N} = \iint n_{\mathbf{k}} d\mathbf{k} = \text{const}$$

$$\boxed{N^{(1D)}_k \sim \zeta^{1/3} k^{-2/3}}$$

finite capacity

Inverse cascade

Phenomenology of GW turbulence

Kinetic equation: $\partial_t n_k = \cancel{\epsilon^2()} + \epsilon^4() + \ldots$ $\quad \epsilon = \dfrac{\tau_{GW}}{\tau_{NL}}$ << 1

$$\Rightarrow \quad \tau_{cascade} \sim \frac{1}{\epsilon^4}\tau_{GW} \sim \left(\frac{\tau_{NL}}{\tau_{GW}}\right)^3 \tau_{NL}$$

$$\tau_{GW} \sim 1/\omega$$
$$\tau_{NL} \sim \ell/h_c$$
$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

Energy density: $E \sim \dfrac{c^4}{32\pi G}\dfrac{h^2}{\ell^2} \sim \omega N$

[Maggiore, 2008]

1D energy and wave action spectra:

$$\varepsilon \sim \frac{E}{\left(\frac{\tau_{NL}}{\tau_{GW}}\right)^3 \tau_{NL}} \sim \frac{E}{\left(\frac{\ell}{h}\right)^4 \omega^3} \sim \frac{E^3}{k^3} \sim E_k^3 \quad \Rightarrow \quad \boxed{E_k \sim \varepsilon^{1/3}}$$

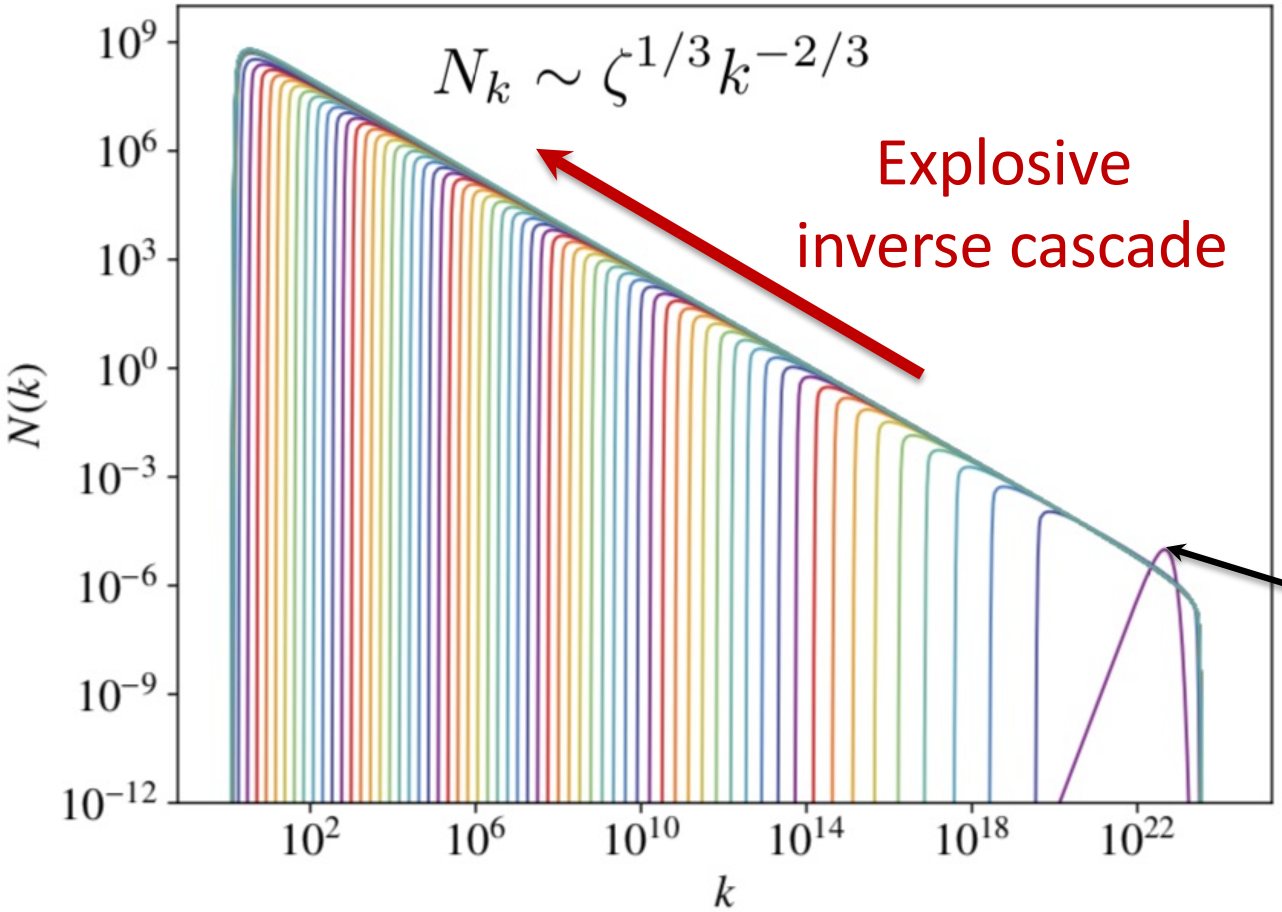
$$\eta \sim \frac{N}{\left(\frac{\tau_{NL}}{\tau_{GW}}\right)^3 \tau_{NL}} \sim \frac{N}{\left(\frac{\ell}{h}\right)^4 \omega^3} \sim \frac{N^3}{k} \sim N_k^3 k^2 \quad \Rightarrow \quad \boxed{N_k \sim \eta^{1/3} k^{-2/3}}$$

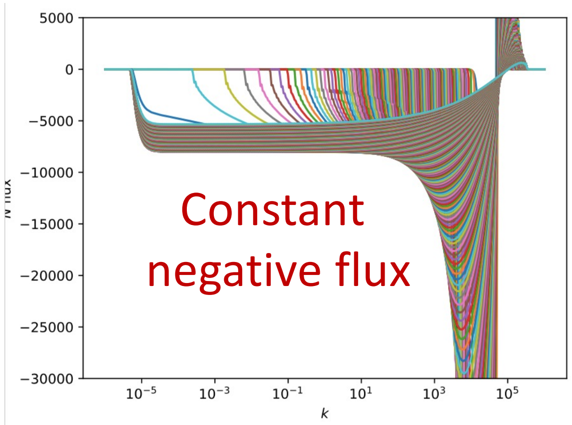
Super-local approximation: nonlinear diffusion model

- Rigorous derivation is rare (in MHD it's possible) [SG & Buchlin, ApJ, 2010]
- Here, it's a phenomenological model

[see also Dyachenko+, Physica D, 1992; Passot & Sulem, JPP, 2019]

$$\frac{\partial N(k)}{\partial t} = \frac{\partial}{\partial k}\left[k^2 N^2(k)\frac{\partial (kN(k))}{\partial k}\right] - \nu k^4 N(k) - \eta \frac{N(k)}{k^4}$$

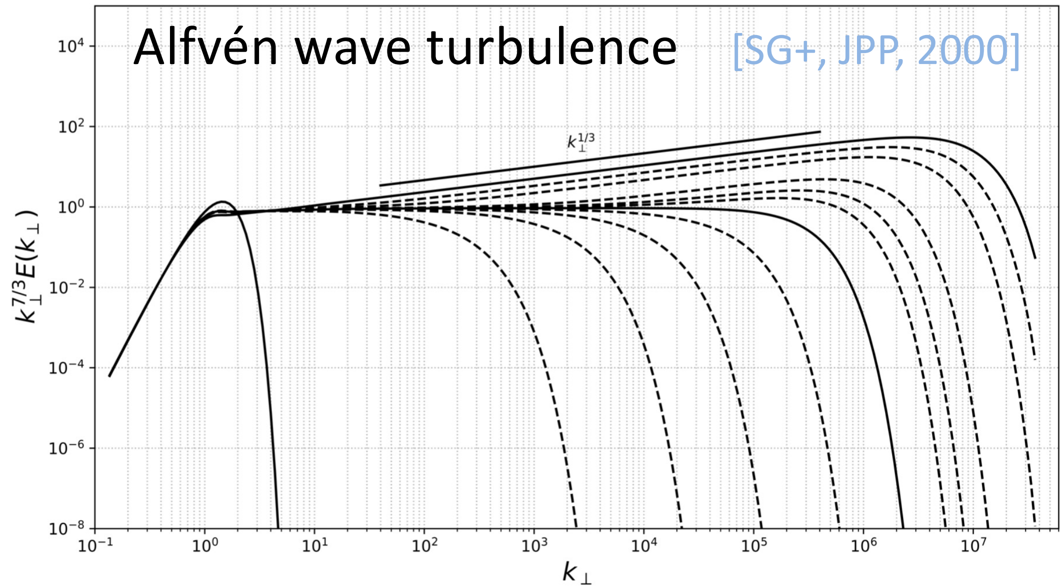




[SG, Nazarenko, Buchlin & Thalabard, Physica D, 2019]

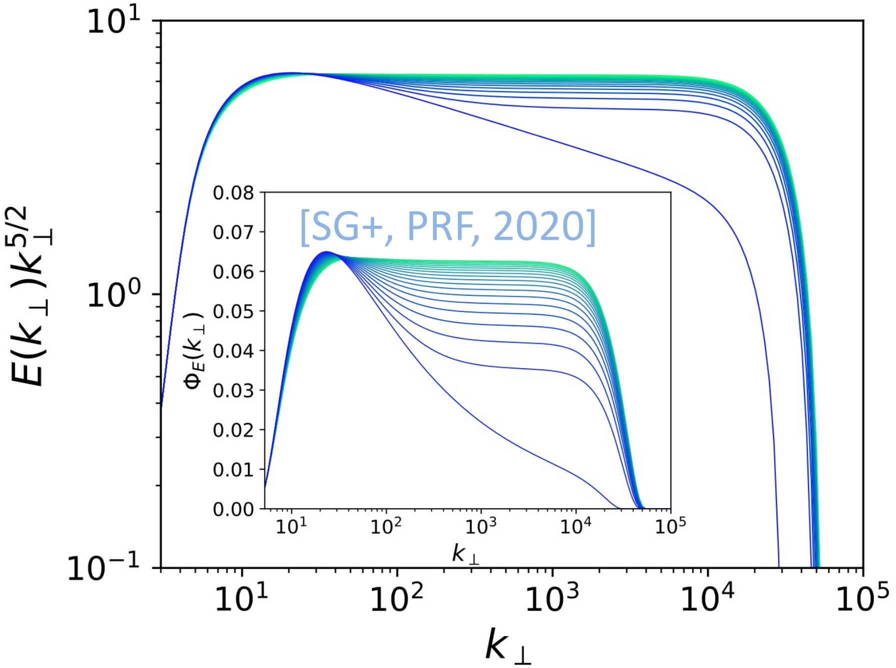
Anomalous scaling

Old subject (weak & strong)

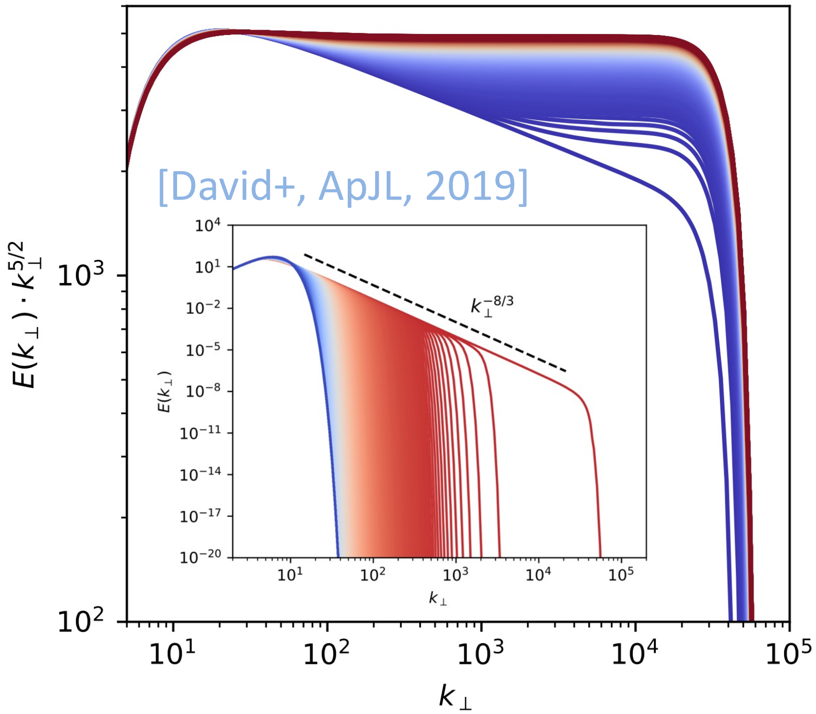


[Semikoz & Tkachev, PRL, 1995; Lacaze+, Physica D, 2001]
[Connaughton & Nazarenko, PRL, 2004; Nazarenko, JETPL, 2006; Boffetta+, JLTP, 2009]
[Thalabard+, JPAMT, 2015]

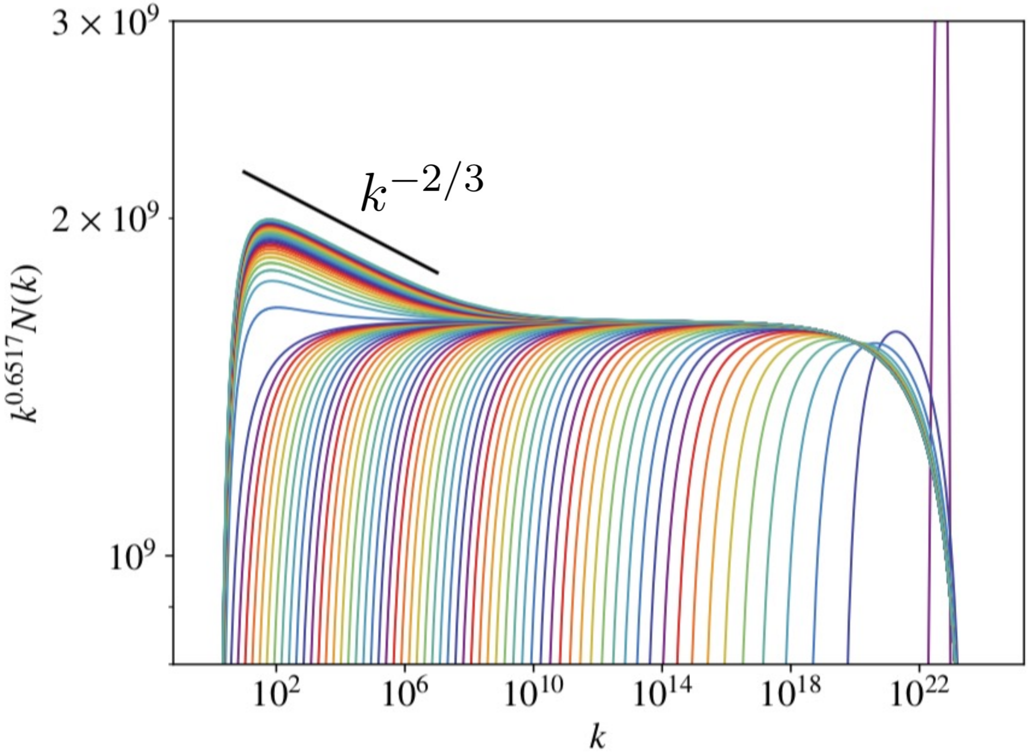
Inertial wave turbulence [SG, PRE, 2003]



Kinetic Alfvén wave turbulence



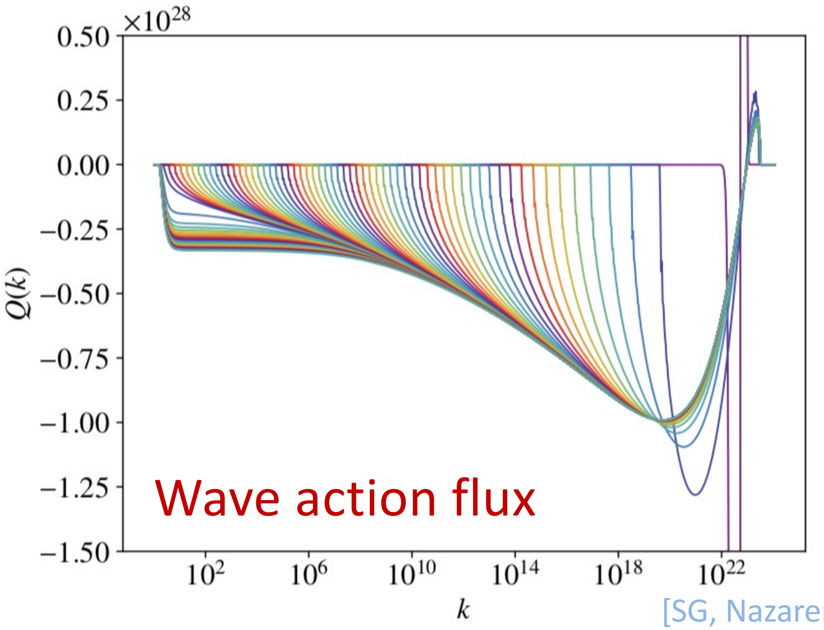
Anomalous scaling

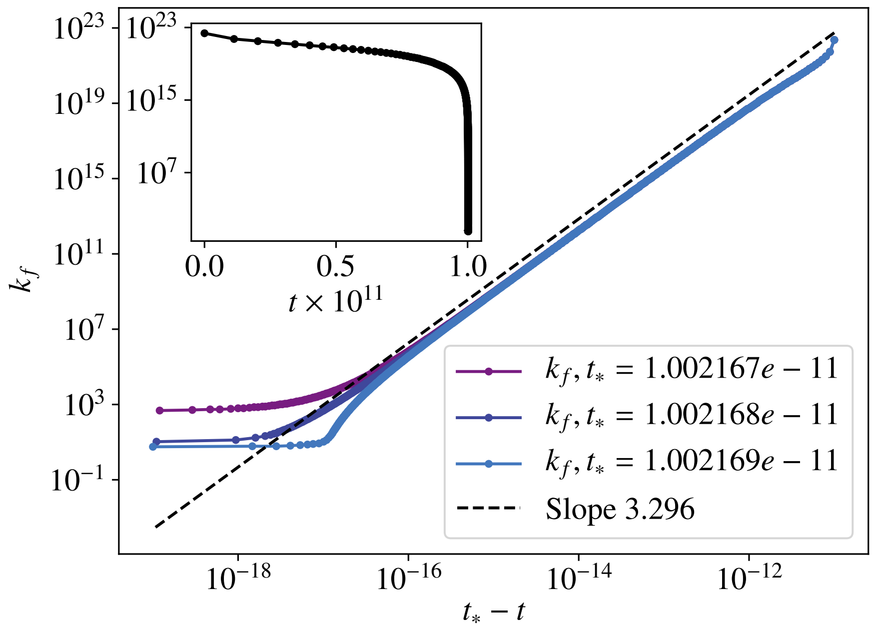


$$\frac{\partial N(k)}{\partial t} = \frac{\partial}{\partial k}\left[k^2 N^2(k)\frac{\partial(kN(k))}{\partial k}\right] - \nu k^4 N(k) - \eta\frac{N(k)}{k^4}$$

Self-similar solution of the second kind:

$$N = \frac{1}{\tau^\alpha}N_0\left(\frac{k}{\tau^\beta}\right) \qquad \tau = t_* - t$$





[SG, Nazarenko, Buchlin & Thalabard, Physica D, 2019]

$$k_f \sim (t_* - t)^{3.296}$$

The first direct numerical simulation of GW turbulence

- Pseudo-spectral code (FFTW3), periodic conditions, de-aliasing
- Adams-Bashforth (explicit; order 2) numerical scheme
- No forcing (decaying turbulence); additional dissipation $-\nu k^4 \dot{\lambda}_{\mathbf{k}}$ (for $k \geq k_{\text{diss}}$)
- Initial condition: $k_{min} << k_i << k_{max}$
- Resolutions 512^2

$$\boxed{\begin{gathered}\partial_x \dot{\tilde{\alpha}} = -2\dot{\lambda}(\partial_x \lambda), \quad \partial_y \dot{\tilde{\beta}} = -2\dot{\lambda}(\partial_y \lambda) \quad \partial_x \partial_y \tilde{\gamma} = -2(\partial_x \lambda)(\partial_y \lambda) \\ \partial_t[(1+\tilde{\alpha}+\tilde{\beta}-\tilde{\gamma})\dot{\lambda}] = \partial_x[(1+\tilde{\alpha}-\tilde{\beta}+\tilde{\gamma})\partial_x \lambda] + \partial_y[(1-\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})\partial_y \lambda]\end{gathered}}$$

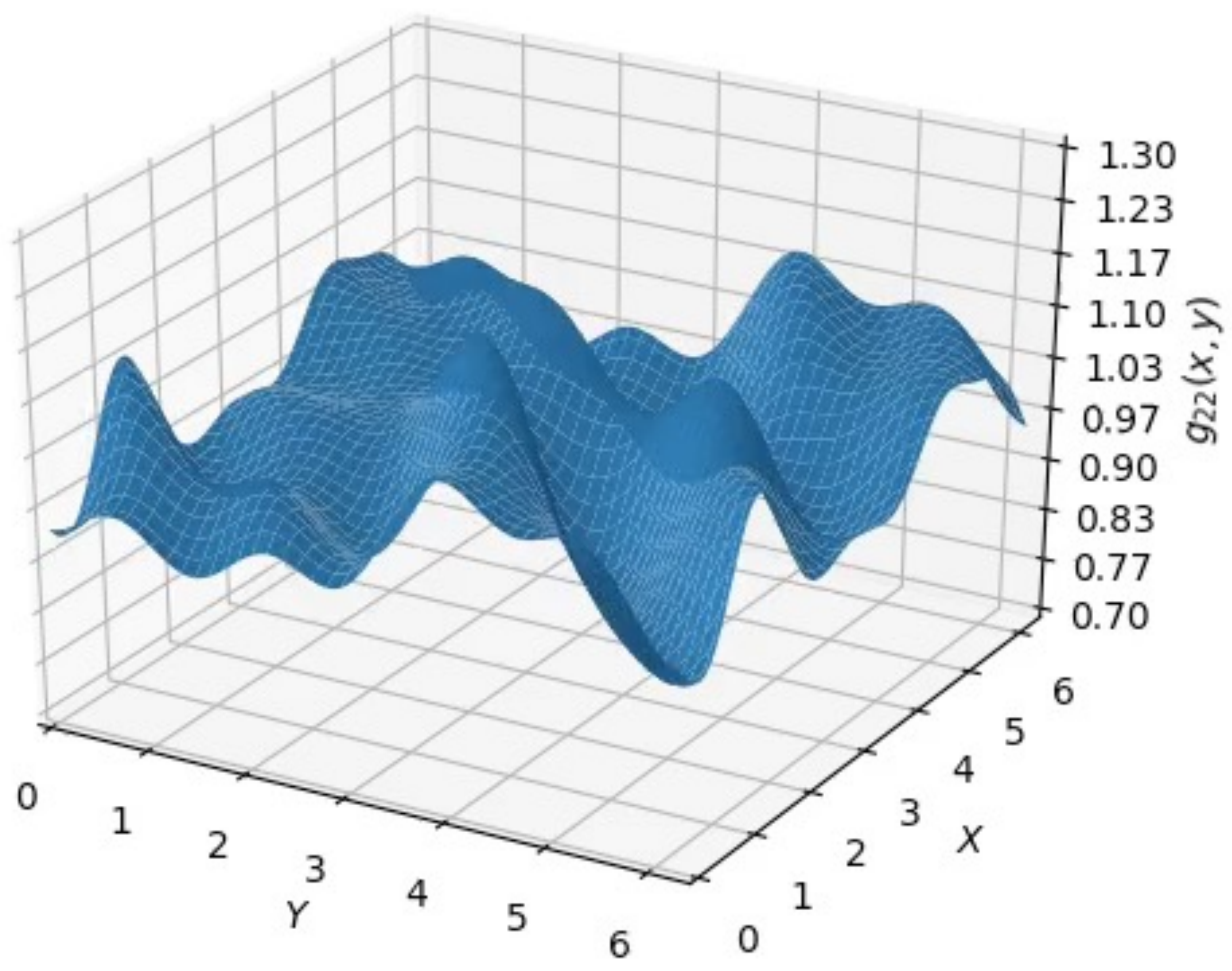
+ intermediate variables: $\Lambda = \partial_t \lambda \quad A = \partial_t \tilde{\alpha},\ B = \partial_t \tilde{\beta},$ and $G = \partial_t \tilde{\gamma}$

$$g_{00} = -(1+\tilde{\gamma})^2 e^{-2\lambda} \qquad g_{22} = (1+\tilde{\alpha})^2 e^{-2\lambda}$$

$$g_{11} = (1+\tilde{\beta})^2 e^{-2\lambda} \qquad g_{33} = e^{2\lambda}$$

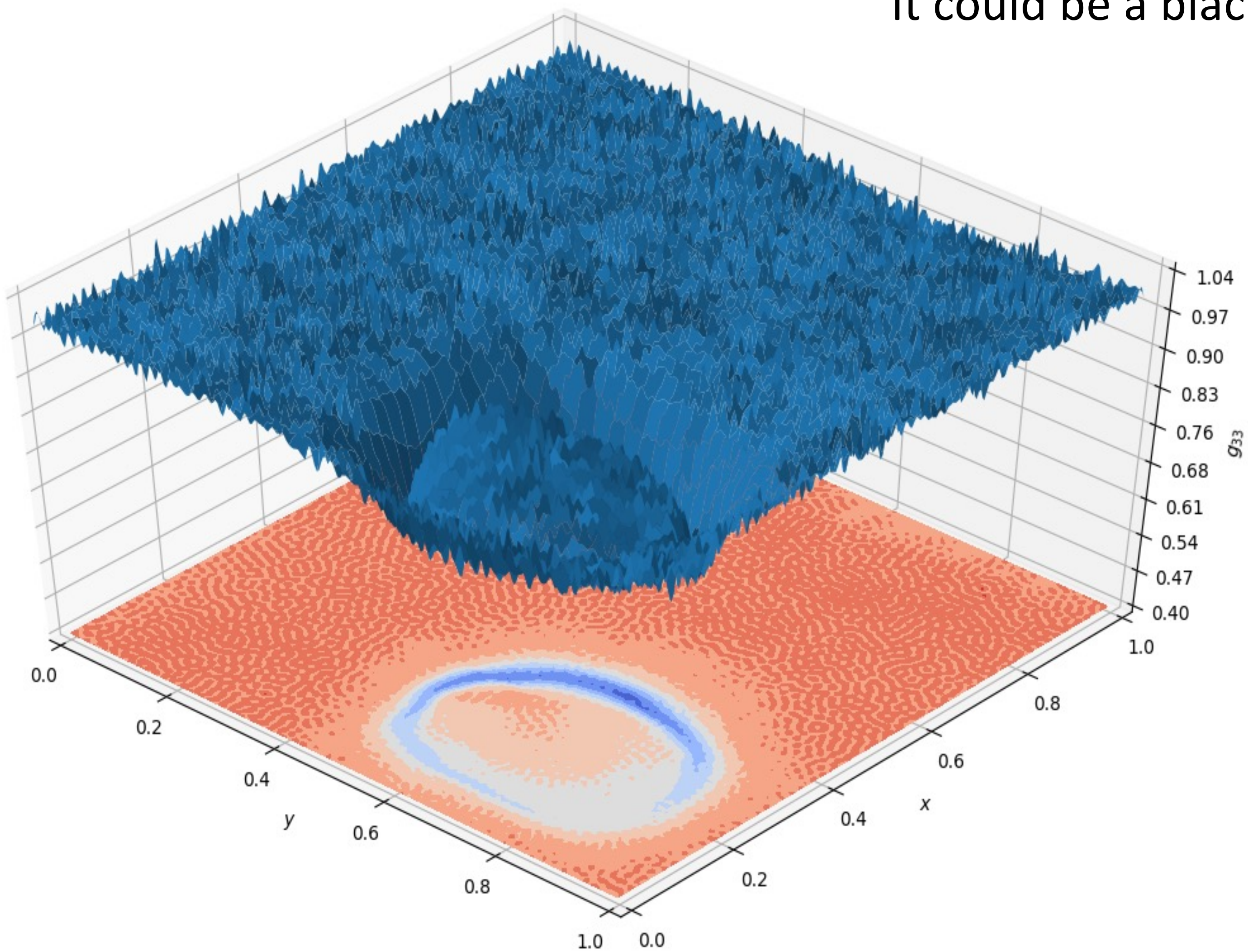
$$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \lambda \ll 1$$

Illustration with an excitation at low k

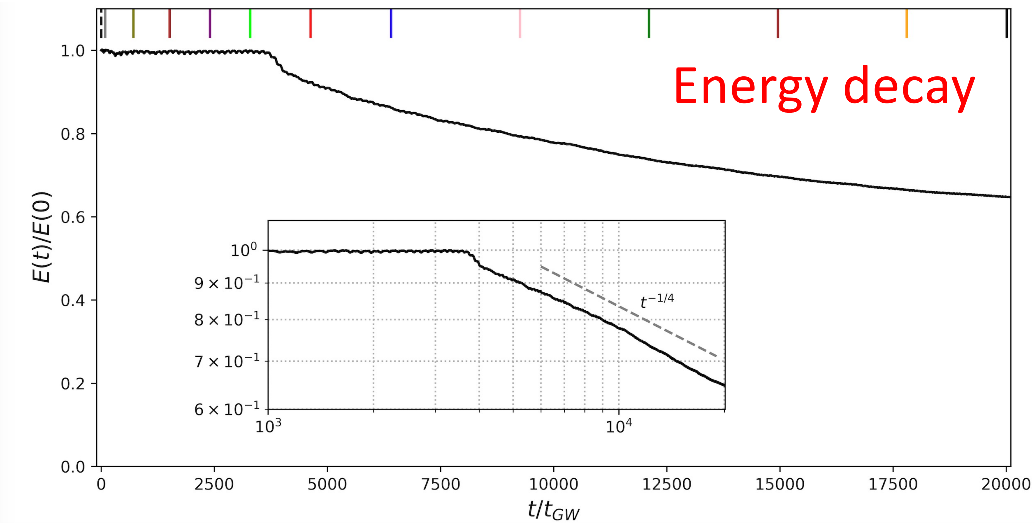


Be careful with the initial condition !

It could be a black hole...



Direct evidence of a dual cascade

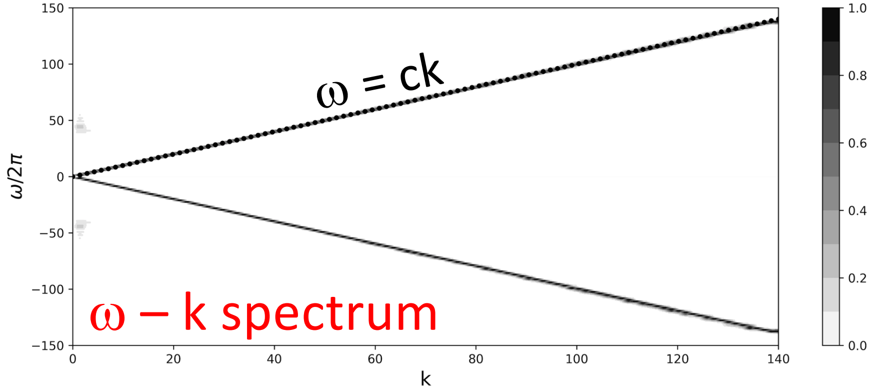


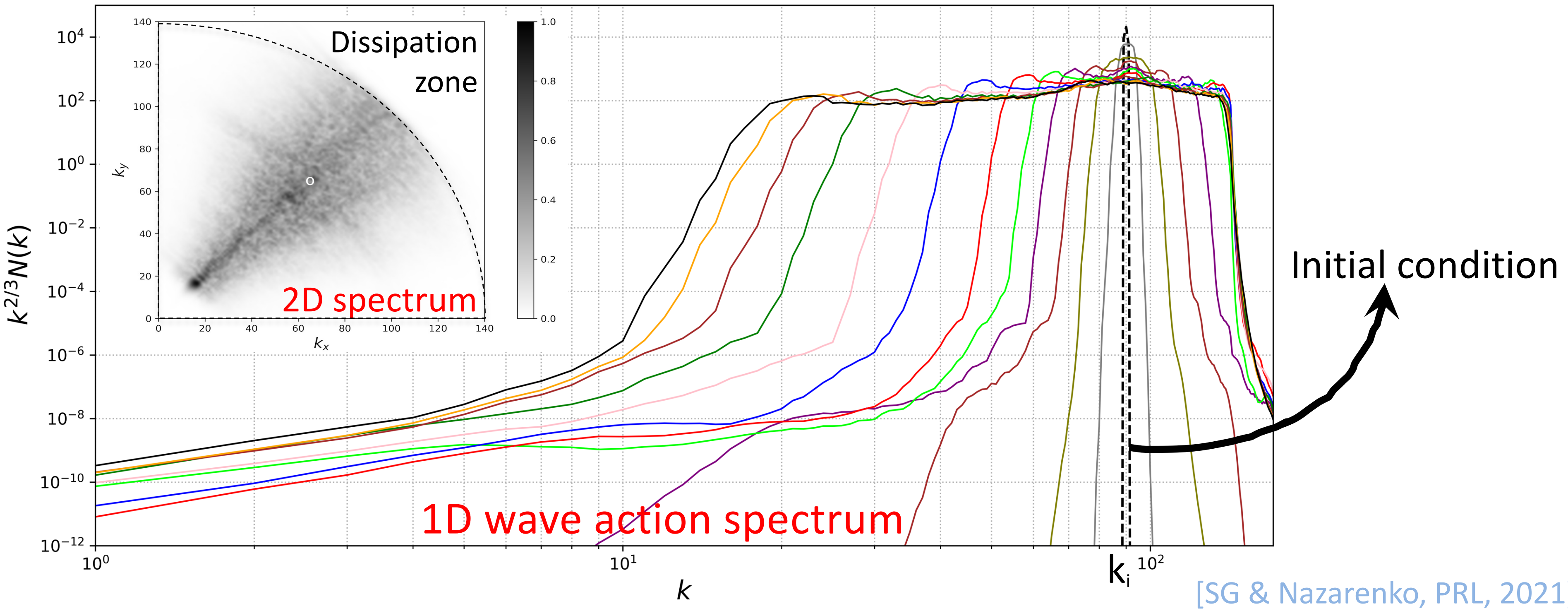
$$t_{GW} = 1/\omega_i = 1/k_i$$

Time scale compatible with

4-wave interactions $\tau_{tr} \sim \tau_{GW}/\epsilon^4$

Weak wave turbulence

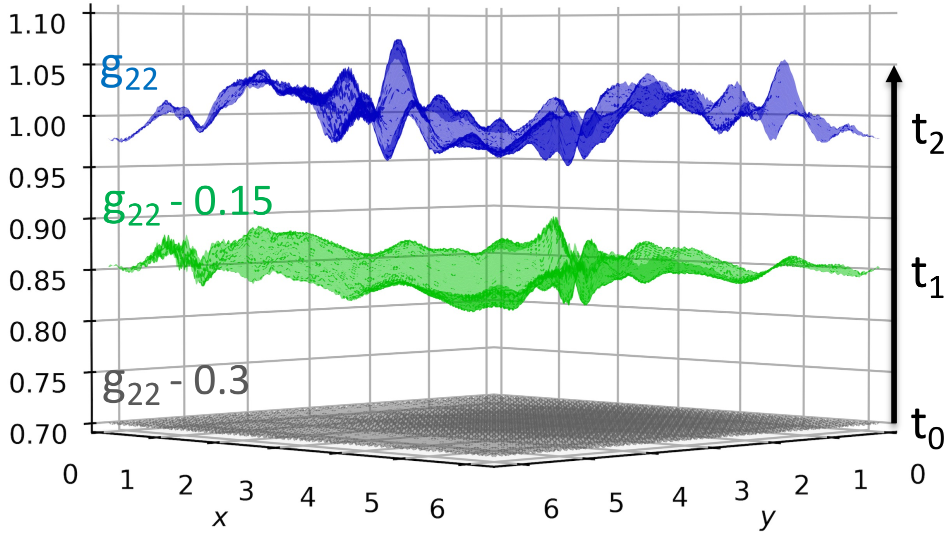




[SG & Nazarenko, PRL, 2021]

Space-time metric

Time evolution of the metric component g_{22}



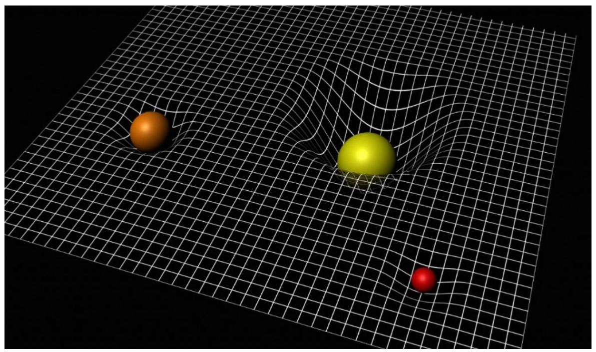
$$g_{\mu\nu} = \begin{pmatrix} -(H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{pmatrix}$$

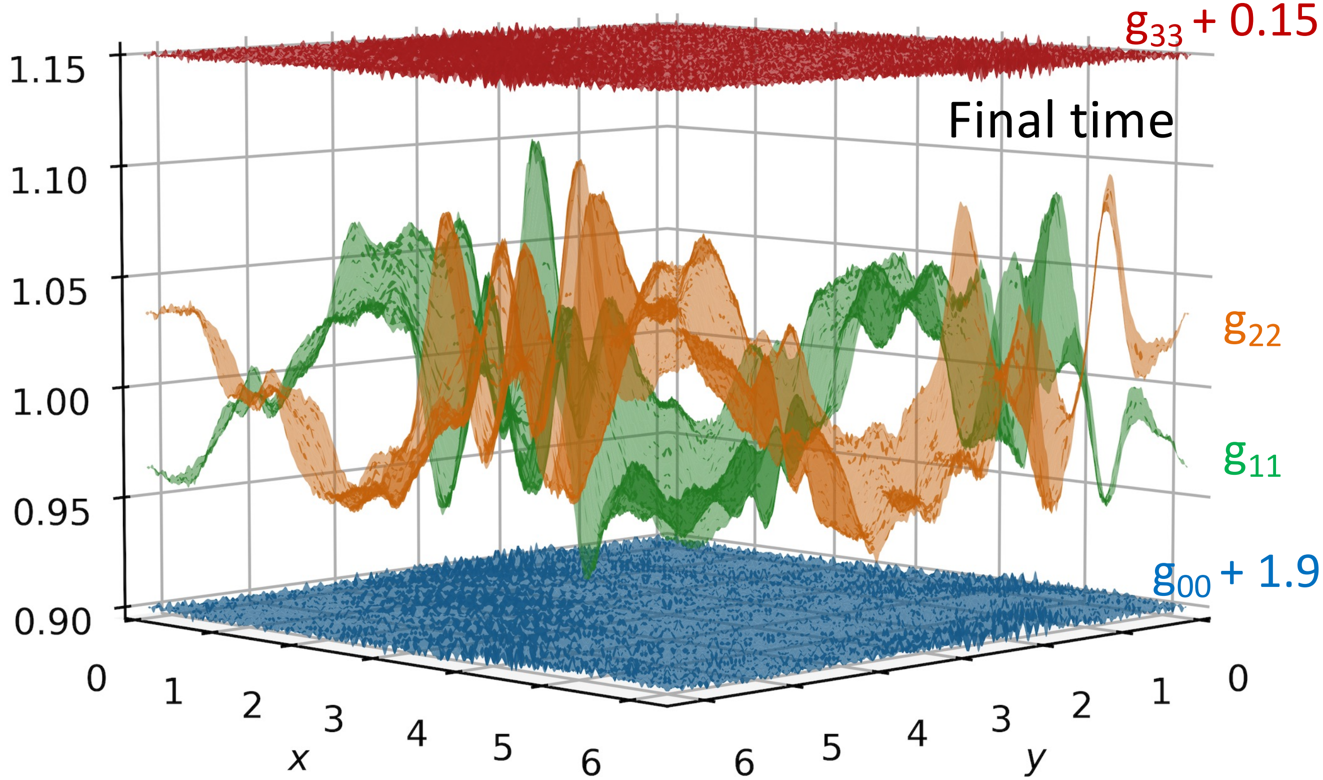
$$H_0 = e^{-\lambda}\gamma, \;\; H_1 = e^{-\lambda}\beta, \;\; H_2 = e^{-\lambda}\alpha, \;\; H_3 = e^{\lambda}$$

Hadad-Zakharov metric

[SG & Nazarenko, PRL, 2021]

Selective amplification





Conclusion and (exciting) perspectives

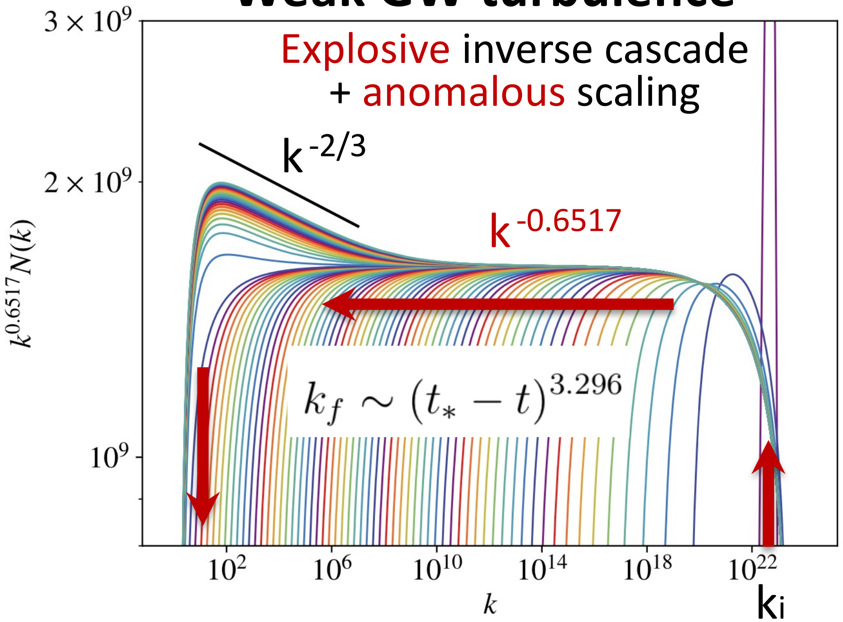
Turbulence in general relativity exists (4-wave interactions)

- GW turbulence is characterized by a *dual* cascade
- *Explosive* inverse cascade of wave action / *anomalous* scaling
- *Strong* GW turbulence is expected at large scale

Beyond weak wave turbulence

From weak to strong GW turbulence

Weak GW turbulence



$$\chi \sim \frac{\tau_{GW}}{\tau_{NL}} \sim h_\ell \sim \ell^{1/3} \ll 1$$

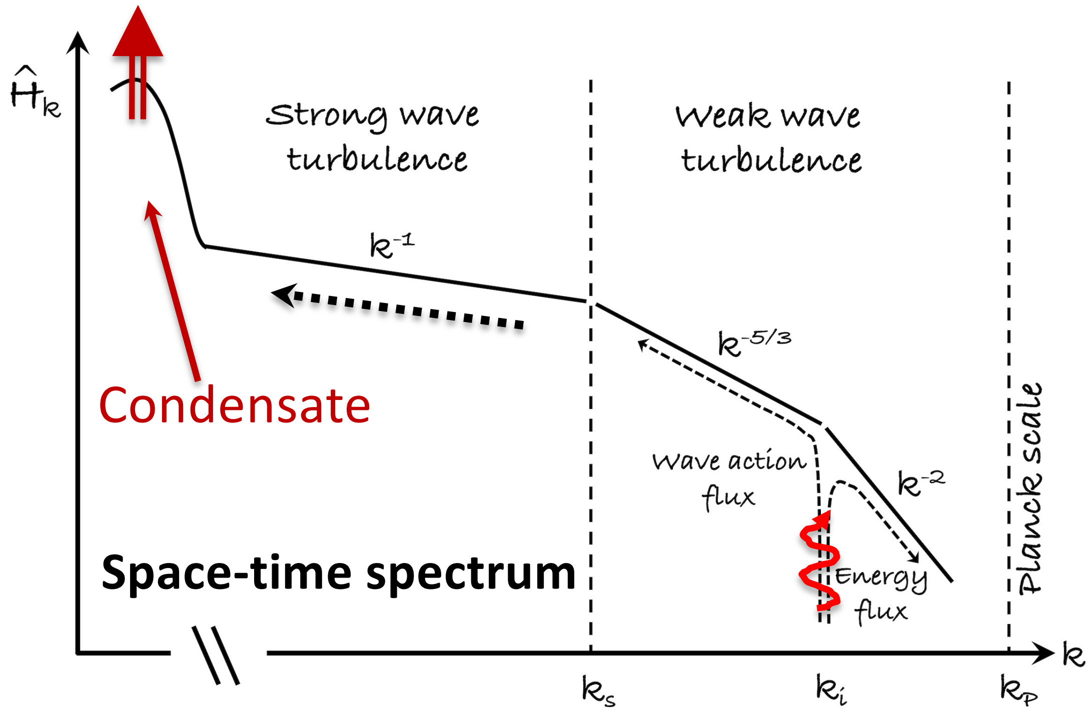
$$\tau_{GW} \sim 1/\omega$$
$$\tau_{NL} \sim \ell/(h_\ell c)$$

Phenomenological model of strong GW turbulence

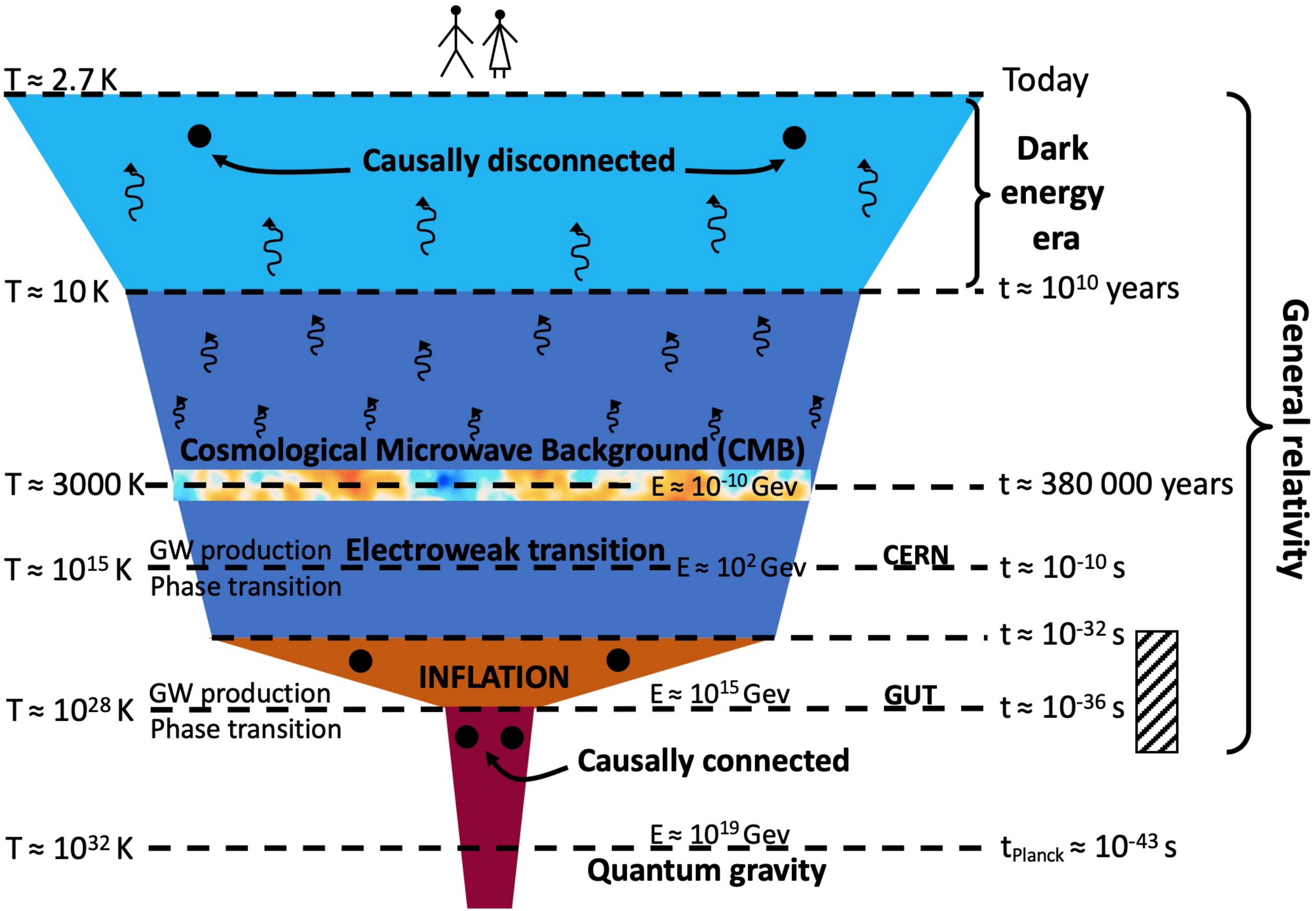
$$\chi \sim \frac{\tau_{GW}}{\tau_{NL}} \sim h_\ell \sim 1$$ critical balance

A fast growth of the condensate = inflation

[SG, Laurie & Nazarenko, Universe, 2020]



History of the observable universe



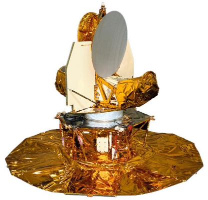
Inflation is not explained from first principles (hypothetical *inflaton*)

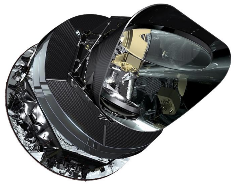
[Ijjas+, PLB, 2013]

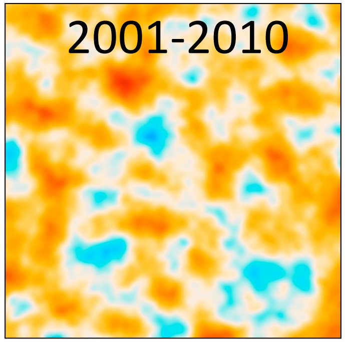
Presence of inhomogeneities

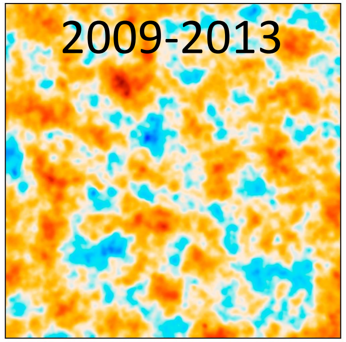
The Cosmic Microwave Background (CMB) is homogeneous

[Penzias & Wilson, ApJ, 1965]







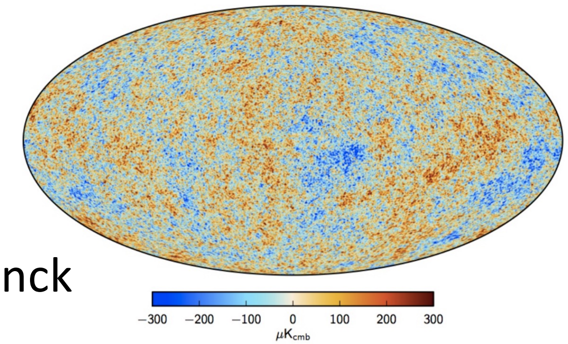


T=2.72548 ± 0.00057 K

Homogeneous up to 1/10 000

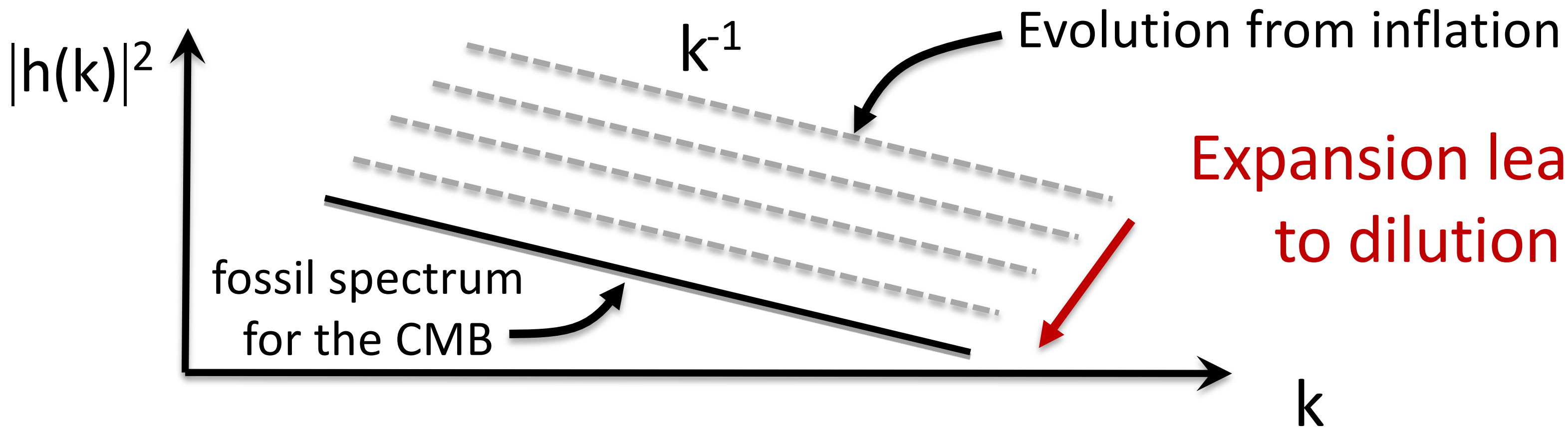
COBE WMAP Planck

Inhomogeneities **are necessary** to explain the structures in the universe



Planck

Comparison with observations



Small fluctuations are treated in the Newtonian limit: $\nabla^2\phi = 4\pi G\rho$

$$E_\ell \sim \frac{c^4}{32\pi G}\frac{h_\ell^2}{\ell^2} \Rightarrow \boxed{P_\phi(k) = \phi^2(k) \sim k^{n_s-2} \sim k^{-1}}$$

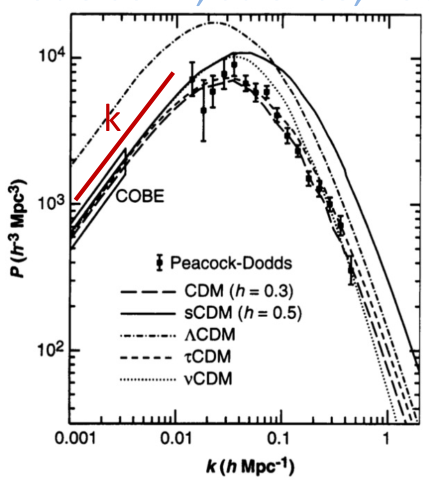
Prediction compatible with the

Harrison-Zeldovich spectrum (n_s=1)

[Harrison, PRD, 1970; Zeldovich, MNRAS, 1972]

$P_\rho(\mathbf{k}) = \rho^2(\mathbf{k}) \sim \rho^2(k)/k^2 \sim k^{n_s} \sim k$

[Dodelson+, science, 1996]

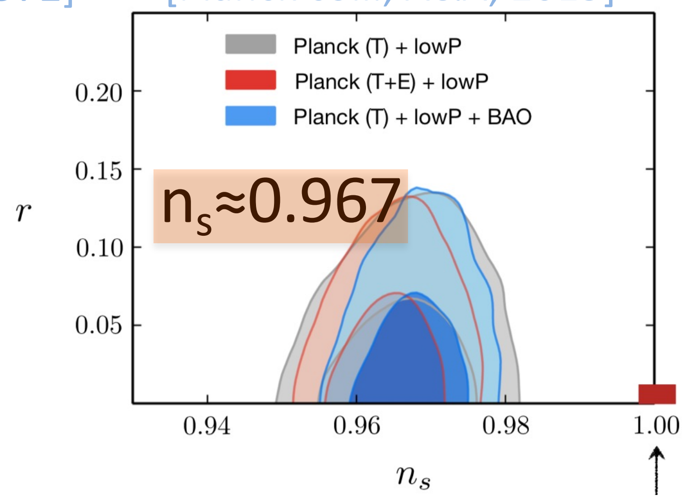


($\delta T/T \sim \delta\rho/\rho \sim 10^{-5}$)

Planck is compatible with the fossil spectrum $|h(k)|^2 \sim k^{-1.033}$

Latest data

[Planck coll., A&A, 2015]



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Towards a paradigm shift for the theory of cosmological inflation ?

- Phenomenological (critical balance) model of *inflation* ($t \geq 10^{-36}$s)
- Fossil spectrum *compatible* with CMB (Planck satellite data)
- *Falsifiable* predictions with DNS (no tuning parameter)

- Problem close to elastic wave turbulence [Hassaini+, 2019]
  - *The Riemann (4th order) curvature tensor and the Kretschmann scalar are non trivial*