Kolmogorov-Zakharov Spectra via the Zakharov Transform

Examples class - Waves And Complexity Summer School
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0 Notation reminder and warm-up exercises

0.1 Notational shorthand

Wave turbulence (WT) is "the study of statistical ensembles of nonlinearly interacting waves". For example in a 4-wave process the Fourier mode $a(\mathbf{k})$ will evolve due to interactions with all other modes $a(\mathbf{k}_1), a(\mathbf{k}_2)$ and $a(\mathbf{k}_3)$, whose wavenumbers satisfy the wavevector resonance condition $\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2 = 0$. (Of course these Fourier modes are functions of time as well but we frequently suppress writing the t dependence.) To compactify the notation we often write the functional dependence on wavevectors as indices, as well as other shortcuts.

For example the 4-wave process described above is governed by the Hamiltonian

$$H_4 = \sum_{1234} W_{12}^{34} \delta_{12}^{34} a_1 a_2 a_3^* a_4^*. \tag{1}$$

Note the shorthand we have used:

$$\begin{aligned}
\sum_{1234} \dots =& \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} \dots && \text{(sum over wavevectors)}\\
W_{12}^{34} =& W(\mathbf{k}_3, \mathbf{k}_4; \mathbf{k}_1, \mathbf{k}_2) && \text{(4-wave interaction coefficient)}\\
\delta_{12}^{34} =& \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1 - \mathbf{k}_2) && \text{(Kronecker delta on wavevectors)}\\
a_i =& a(\mathbf{k}_i) && \text{(Fourier mode).}
\end{aligned}$$

Much of the work of these examples classes is made easier by familiarising yourself with this notation. In fact most of these exercises involve tricks to do with the manipulation of dummy indices in sums like (1), and dummy integration variables in integrals like (4). I invite you to practice this in the following exercise.

0.1.1 Exercise: symmetries of the 4-wave interaction coefficient

By considering equation (1) for the quartic Hamiltonian, show that the 4-wave interaction coefficient has the following symmetries:

$$W_{12}^{34} = W_{12}^{43} = W_{21}^{34} = \left(W_{34}^{12}\right)^* \tag{2}$$

(for the last of these you will need to remember that the Hamiltonian is a functional whose numerical value is an energy. What kind of number can represent an energy?)

0.1.2 Exercise: drawing a resonant quartet

To start visualising the modes involved in a 4-wave interaction, draw a quartet of waves that satisfy the wavevector resonance condition embodied by $\delta_{12}^{\mathbf{k}3}$.

0.1.3 Exercise: scaling of Dirac delta function

Show, by an appropriate change of variable and using the defining property of the Dirac delta function $\int_{\mathbb{R}} dx f(x)\delta(x) = f(0)$, that

$$\delta(Ax) = \frac{1}{|A|}\delta(x). \tag{3}$$

(Pay attention to whether and when the sign of A induces a swap of the integration limits.)

1 Kolmogorov-Zakharov spectra via the Zakharov transform

Our task here is to derive the WT Kolmogorov-Zakharov (KZ) flux spectra, starting from a fairly general 4-wave kinetic equation.

In the WT limit the 4-wave interaction process outlined in section 0.1 results in the wave kinetic equation (WKE)

$$\dot{n}_{\mathbf{k}} = 4\pi \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \left|W_{12}^{\mathbf{k}3}\right|^2 \delta_{12}^{\mathbf{k}3}\, \delta(\omega_{12}^{\mathbf{k}3}) n_1 n_2 n_3 n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_3} - \frac{1}{n_1} - \frac{1}{n_2}\right] \tag{4}$$

The right-hand side of (4) is frequently referred to as the collision integral Coll$[n]$. More notation to understand:

- $\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \dots$ denotes 3 integrals over d-dimensional $\mathbf{k}$-space,
- $\delta_{12}^{\mathbf{k}3}$ is now a Dirac delta describing wavenumber resonance,
- $\delta(\omega_{12}^{\mathbf{k}3}) = \delta(\omega_{\mathbf{k}} + \omega_3 - \omega_1 - \omega_2)$ is a Dirac delta describing frequency resonance,
- $\omega_i = \omega(\mathbf{k}_i)$ is the frequency of waves with wavenumber $\mathbf{k}_i$, i.e. the dispersion relation,
- $n_i = n(\mathbf{k}_i)$ is the waveaction spectrum at wavenumber $\mathbf{k}_i$.

The WKE conserves the two invariants waveaction $N = \int d\mathbf{k}\, n_{\mathbf{k}}$ (a.k.a. particle number or mass), and energy $E = \int d\mathbf{k}\, \omega_{\mathbf{k}} n_{\mathbf{k}}$. The KZ spectra that we will derive today are two scale-invariant (power-law) spectra that are stationary solutions of (4). Although we don't have time to show it today, one of these spectra is responsible for the flux of N to large scale, and the other responsible for the flux of E to small scale. (The fact that WT kinetic equations have these scale-invariant flux spectra is strongly reminiscent of Kolmogorov's 1941 theory of hydrodynamic turbulence, and is indeed the reason why this field is called wave *turbulence*.)

Sketch of derivation

We will find stationary solutions of (4) by writing transforming the RHS into

$$\int dk_1 dk_2 dk_3\, (k_1 k_2 k_3)^{d-1} n_1 n_2 n_3 n_k\, T_{12}^{\mathbf{k}3} \left[1 + \left(\frac{k_3}{k}\right)^x - \left(\frac{k_1}{k}\right)^x - \left(\frac{k_2}{k}\right)^x\right], \tag{5}$$

and then finding the values of x that make the $[\dots]$ vanish.

The steps we will take are:

1.1 Assume isotropy of system. Collect angular dependence and symmetrisable part of Coll$[n]$ into $T_{12}^{\mathbf{k}3}$.

1.2 Establish symmetry properties $T_{12}^{\mathbf{k}3} = T_{21}^{\mathbf{k}3} = -T_{\mathbf{k}2}^{13} = -T_{1\mathbf{k}}^{23}$.

1.3 Assume scaling properties of n_k, ω_k, and $W_{12}^{\mathbf{k}3}$. Apply Zakharov transformation to get k-scalings of each tem in Coll$[n]$, and establish eq. (5).

Let's start the derivation!

1.1 Assume isotropy and collect isotropic and angle-dependent parts of Coll$[n]$

If the system we are considering is isotropic then no function of a single $\mathbf{k}$ depends on its direction, only on its magnitude k. In particular $n_{\mathbf{k}} = n_k$ and $\omega_{\mathbf{k}} = \omega_k$. However functions of multiple wavevectors may depend on their relative orientation.

1.1.1 Exercise: consider each factor in Coll$[n]$

Examine each part of eq. (4), assuming $n_{\mathbf{k}} = n_k$ and $\omega_{\mathbf{k}} = \omega_k$. Convince yourself that

- $\delta(\omega_{12}^{k3}) = \delta(\omega_k + \omega_3 - \omega_1 - \omega_2)$ does not depend on the relative orientations between $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and $\mathbf{k}$.
- However, $\delta_{12}^{\mathbf{k}3}$ *does* depend on the relative orientations of the $\mathbf{k}_i$'s. In particular:

$$\begin{aligned}\delta_{12}^{\mathbf{k}3} &= \delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) \\ &= \underbrace{\delta(k_x + k_{3x} - k_{1x} - k_{2x})\,\delta(k_y + k_{3y} - k_{1y} - k_{2y})\ldots}_{d \text{ components}} \\ &\overset{2d}{=} \delta(k + k_3\cos\theta_3 - k_1\cos\theta_1 - k_2\cos\theta_2)\delta(k_3\sin\theta_3 - k_1\sin\theta_1 - k_2\sin\theta_2)\end{aligned} \tag{6}$$

  where the last line is for the $d{=}2$ case and angles are measured relative to the direction of $\mathbf{k}$ (**Exercise:** visualise this by going back to your drawing in Exercise 0.1.2 and marking the angles θ_1, θ_2, θ_3).

- $|W_{12}^{\mathbf{k}3}|^2$ in general depends on the relative orientations of its wavevector arguments (this is not the case for the nonlinear Schrödinger equation for which $|W_{12}^{\mathbf{k}3}|^2 = 1$).
- $\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 = \int_0^\infty dk_1 \int_0^\infty dk_2 \int_0^\infty dk_3 (k_1k_2k_3)^{d-1} \int_{S_1} dS_1 \int_{S_2} dS_2 \int_{S_3} dS_3$. Here dS_i is an area element of the $(d-1)$ unit sphere S_i (the area of S is 2π in $d{=}2$ and 4π in $d{=}3$). Yet more shorthand: let us write $dk_1dk_2dk_3 = dk_{123}$ and $dS_1dS_2dS_3 = dS_{123}$.

1.1.2 Exercise: collecting terms in the WKE

Further convince yourself that you can rewrite Coll$[n]$ as

$$\text{Coll}[n] = \int_0^\infty dk_{123}\,(k_1k_2k_3)^{d-1}\, n_1n_2n_3n_kT_{12}^{\mathbf{k}3} \tag{7}$$
$$\text{where } T_{12}^{\mathbf{k}3} = 4\pi\left[\frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_1} - \frac{1}{n_2}\right]\delta\left(\omega_{12}^{k3}\right)\int dS_{123}\delta_{12}^{\mathbf{k}3}|W_{12}^{\mathbf{k}3}|^2.$$

1.2 Symmetrise eq. (7)

1.2.1 Exercise: symmetry properties of $T_{12}^{\mathbf{k}3}$

Show that $T_{12}^{\mathbf{k}3}$ has the symmetries

$$T_{12}^{\mathbf{k}3} = T_{21}^{\mathbf{k}3} = -T_{\mathbf{k}2}^{13} = -T_{1\mathbf{k}}^{23} \tag{8}$$

Using (8) we split (7) into four equal parts, and rewrite it as

$$\mathrm{Coll}[n] = \frac{1}{4}\int dk_{123}\, n_1 n_2 n_3 n_k \,(k_1 k_2 k_3)^{d-1} \,[\, \underbrace{T^{\mathbf{k}3}_{12}}_{(A)} + \underbrace{T^{\mathbf{k}3}_{21}}_{(B)} - \underbrace{T^{13}_{\mathbf{k}2}}_{(C)} - \underbrace{T^{23}_{1\mathbf{k}}}_{(D)}\,] \tag{9}$$

1.3 Zakharov transformation

We now consider each piece (A)–(D) in (9) seperately. Part (A) we leave alone. We can map parts (B), (C), and (D) onto part (A) by using the appropriate Zakharov transform (ZT).

The Zakharov transform for part (B) is a transformation of the integration variables $(k_1, k_2, k_3) \to (\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$ given implicitly by the following:

$$k_1 = \tilde{k_1}\frac{k}{\tilde{k}_3} \quad , \quad k_2 = \tilde{k}_2\frac{k}{\tilde{k}_3} \quad , \quad k_3 = \frac{k^2}{k}\left(= k\frac{k}{\tilde{k}_3}\right). \tag{10}$$

It is also useful to note that identically $k = \tilde{k}_3 k/\tilde{k}_3$; observe that in the ZT the modulus of each wavevector is being scaled by $k/\tilde{k}_3$.

(Notice that the k_3 transform is non-identity, in particular

$$\int_l^u dk_3 \xrightarrow{ZT} \int_{\tilde{u}}^{\tilde{l}} d\tilde{k}_3, \tag{11}$$

i.e. this part of the ZT exchanges the upper and lower limits. This will be important when considering whether the KZ spectra found via the ZT is local.)

Our final set of assumptions is that we have a scale-invariant spectrum, and that the dispersion relation and interaction coefficient scale homogeneously with their arguments. Namely, we assume:

- scale-invariant spectrum $n_k = Ak^\nu$,
- scale-invariant dispersion relation $\omega_k = \lambda k^\alpha$ (homogeneous with degree α),
- $W^{\mathbf{k}3}_{12}$ to be homogeneous with degree β, i.e. $W^{\mu\mathbf{k}\,\mu\mathbf{k}_3}_{\mu\mathbf{k}_1\,\mu\mathbf{k}_2} = \mu^\beta W^{\mathbf{k}3}_{12}$.

We are now in a position to apply the ZT to each part in (B), which is

$$\int dk_{123}\, n_1 n_2 n_3 n_k \,(k_1 k_2 k_3)^{d-1} T^{\mathbf{k}3}_{21} \tag{12}$$

We take each part in turn in the following series of exercises.

1.3.1 Exercise: dk_{123}

Show that

$$dk_{123} \overset{ZT}{=} \left(\frac{\tilde{k}_3}{k}\right)^{-4} d\tilde{k}_{123}.$$

Hint: recall that the Jacobian of a coordinate transform is given by $J = \left|\det\left(\frac{\partial k_j}{\partial \tilde{k}_i}\right)\right|$. By taking the absolute value of the determinant, we assert that any changes of orientation of the integral induced by the ZT are accounted for in the limits of the integral; see eq. (11) , and Exercise 0.1.3).

1.3.2 Exercise: $n_1 n_2 n_3 n_k \left(k_1 k_2 k_3\right)^{d-1}$

Show that

$$n_1 n_2 n_3 n_k \left(k_1 k_2 k_3\right)^{d-1} \overset{ZT}{=} \tilde{n}_1 \tilde{n}_2 \tilde{n}_3 n_k \left(\tilde{k}_1 \tilde{k}_2 \tilde{k}_3\right)^{d-1} \left(\frac{\tilde{k}_3}{k}\right)^{-4\nu-4d+4}$$

where $\tilde{n}_i = n(\tilde{k}_i)$.

1.3.3 Exercise: $T_{21}^{\mathbf{k}3}$

Now consider

$$T_{21}^{\mathbf{k}3} = 4\pi \left[\frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_2} - \frac{1}{n_1} \right] \delta\left(\omega_{21}^{k3}\right) \int dS_{123} \delta_{21}^{\mathbf{k}3} |W_{21}^{\mathbf{k}3}|^2.$$

Show that:

- $\left[\frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_2} - \frac{1}{n_1} \right] \overset{ZT}{=} \left[\frac{1}{\tilde{n}_k} + \frac{1}{\tilde{n}_3} - \frac{1}{\tilde{n}_1} - \frac{1}{\tilde{n}_2} \right] \left(\frac{\tilde{k}_3}{k} \right)^{\nu}$,

- $\delta\left(\omega_{21}^{k3}\right) \overset{ZT}{=} \delta\left(\omega_k + \tilde{\omega}_3 - \tilde{\omega}_1 - \tilde{\omega}_2\right) \left(\frac{\tilde{k}_3}{k} \right)^{\alpha}$ where $\tilde{\omega}_i = \omega(\tilde{k}_i)$, (Hint: recall exercise 0.1.3)

- $\delta_{21}^{\mathbf{k}3} \overset{ZT}{=} \delta(\mathbf{k} - \tilde{\mathbf{k}}_3 - \tilde{\mathbf{k}}_1 - \tilde{\mathbf{k}}_2) \left(\frac{\tilde{k}_3}{k} \right)^{d}$ (Hint: exercise again 0.1.3 and also recall eq. (6))

- $|W_{21}^{\mathbf{k}3}|^2 \overset{ZT}{=} |\tilde{W}_{12}^{\mathbf{k}3}|^2 \left(\frac{\tilde{k}_3}{k}\right)^{-2\beta}$ (Hint: exercise 0.1.1)

and hence that

$$T_{21}^{\mathbf{k}3} \overset{ZT}{=} \tilde{T}_{12}^{\mathbf{k}3} \left(\frac{\tilde{k}_3}{k}\right)^{\nu+\alpha+d-2\beta}.$$

where $\tilde{T}_{12}^{\mathbf{k}3}$ is a shorthand for $T_{12}^{\mathbf{k}3}$ when its components are themselves functions of the $\tilde{\mathbf{k}}_i$.

1.3.4 Exercise: Mapping of (B) onto (A)

Putting all of the above together, show that (B) defined in (12) becomes

$$\int d\tilde{k}_{123}\, \tilde{n}_1\tilde{n}_2\tilde{n}_3\tilde{n}_k \left(\tilde{k}_1\tilde{k}_2\tilde{k}_3\right)^{d-1} \tilde{T}_{12}^{\mathbf{k}3} \left(\frac{\tilde{k}_3}{k}\right)^{x}$$

where $x = \alpha - 3\nu - 3d - 2\beta$.

Finally, we drop the tildes as the $\tilde{k}_i$ are merely dummy variables. Examine eq. (9) once more. Convince yourself that what you have done is to map (B) onto (A), and picked up a factor of $(k_3/k)^x$ in the process. Please observe how the choice of ZT (10) induced a positional swap of $\mathbf{k} \leftrightarrow \tilde{\mathbf{k}}_3$ in each part of the calculation.

1.3.5 Exercise: Mapping of (C) onto (A)

Part (C) of eq. (9) contains the term $T_{\mathbf{k}2}^{13}$. Consider form of ZT you will need to map this term to (A), which contains $T_{12}^{\mathbf{k}3}$ instead (what indices will you need to swap and what ZT will achieve this?). Convince yourself that once you go through the ZT you will pick up a factor of $(k_1/k)^x$.

1.3.6 Exercise: Mapping of (D) onto (A)

Make the same considerations as in Exercise 1.3.5 and consider how to map (D) to (A). What factor will you pick up?

1.4 Finding the KZ spectra

After all the ZTs we obtain

$$\dot{n}_k = \frac{1}{4}\int dk_1 dk_2 k_3 n_1 n_2 n_3 n_k \left(k_1 k_2 k_3\right)^{d-1} T_{12}^{k3}\left[1+\left(\frac{k_3}{k}\right)^x-\left(\frac{k_1}{k}\right)^x-\left(\frac{k_2}{k}\right)^x\right] \tag{13}$$

where $x = \alpha - 3\gamma - 3d - 2\beta$.

The final step in the calculation is to choose x so that the bracket $[\ldots]$ vanishes. When we do the spectrum n_k will be stationary, and we will have found the KZ spectra.

1.4.1 Particle flux spectrum

One obvious choice is $x=0$, for then we have $[1+1-1-1]=0$. It turns out that this is corresponds to the waveaction flux spectrum

$$n_k = Ak^{\nu_N}, \quad \nu_N = -\frac{2\beta}{3} - \alpha + \frac{\alpha}{3}$$

1.4.2 Energy flux spectrum

The other choice is $x=\alpha$ because $[\ldots] = \left[\frac{k^\alpha + k_3^\alpha - k_1^\alpha - k_2^\alpha}{k^\alpha}\right] = 0$ by virtue of the $\delta(\omega_{12}^{k3})$ sitting inside $T_{12}^{\mathbf{k}3}$. This is the energy flux spectrum

$$n_k = Ak^{\nu_E}, \quad \nu_E = -\frac{2\beta}{3} - d$$

We will not show in this examples class that these spectra indeed correspond to the fluxes of these invariants. For details see e.g. Sergey Nazarenko, *Wave Turbulence*, Springer, 2011.

1.5 Locality of KZ spectra

Using ZTs gives us stationary solutions of the WKE, but there is no guarantee that these solutions are physically meaningful. In particular Coll$[n]$ could diverge when the KZ spectra are substituted back into the original WKE (4). This divergence is masked when carrying out the ZT as the ZT is a non-identity transform: the limits of one of the integral gets swapped, as we saw in eq. (11). The swapping of these limits means that one could have cancellation of divergences when adding and subtracting terms due to the $\left[1 + \left(\frac{k_3}{k}\right)^2 - \ldots - \ldots\right]$ factor.

Even though in this case the KZ spectrum is not a solution of the WKE, it gives us a clue into what physical effect the mathematics is trying to tell us about. The physical interpretation of these diverging KZ spectra is that the flux is non-local (interactions at a given scale k are dominated by far-away scales, i.e. those around the divergence). Do not forget that we assumed the flux was local (interactions at every scale are dominated by nearby scales) when we assumed the scale-free form of the spectrum $n_k = Ak^\nu$.

The upshot of this discussion is that we need to check that the KZ spectra obtained via the ZTs are local by substituting them back into the original WKE and checking that it converges. This is in general not a trivial task!