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Kelvin-wave turbulence

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Collaborators

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Outline

I. Introduction
- Classical vs. quantum turbulence

II. Kelvin Wave Turbulence Theory
- Hamiltonian description, resonant wave interactions, kinetic equations, locality

III. Numerical Simulations
- Biot-Savart and Gross-Pitaevskii equations

Nonlinearity, Complex Phenomena and Universality for Waves
15th-20th May 2022, Ile de Porquerolles, France

Turbulence at Large Scales

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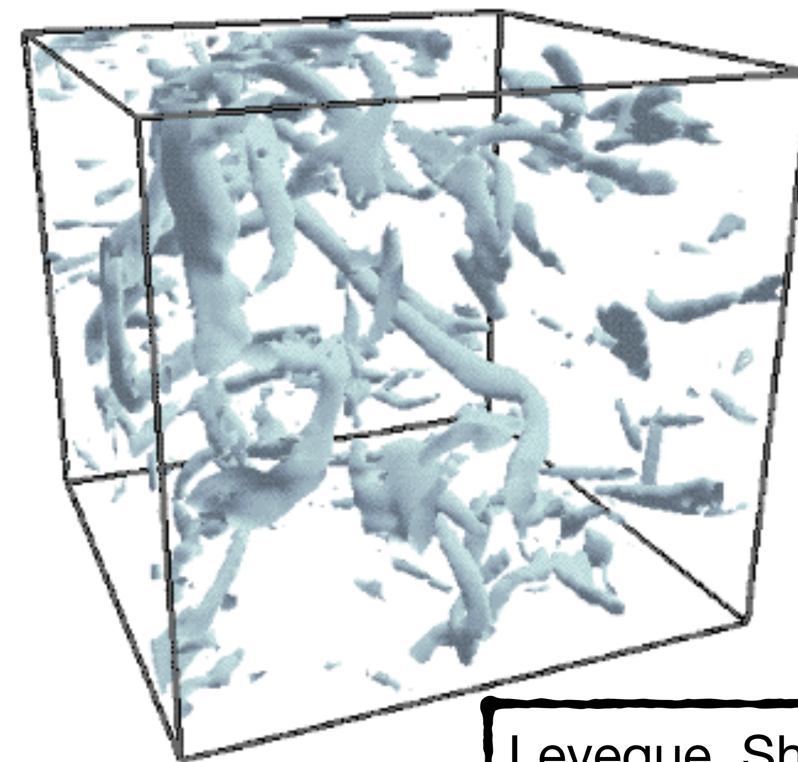
Polarized vortex bundles and K41

- Polarization of quantum vortex lines into bundles mimic vortex tubes of classical turbulence
- Polarization can be induced from large-scale mixing or from normal fluid at finite-temperature
- Similar Richardson cascade scenario of cascading vortex bundles
- Observation of Kolmogorov energy spectrum

Navier-Stokes



Leveque, She, (1993)

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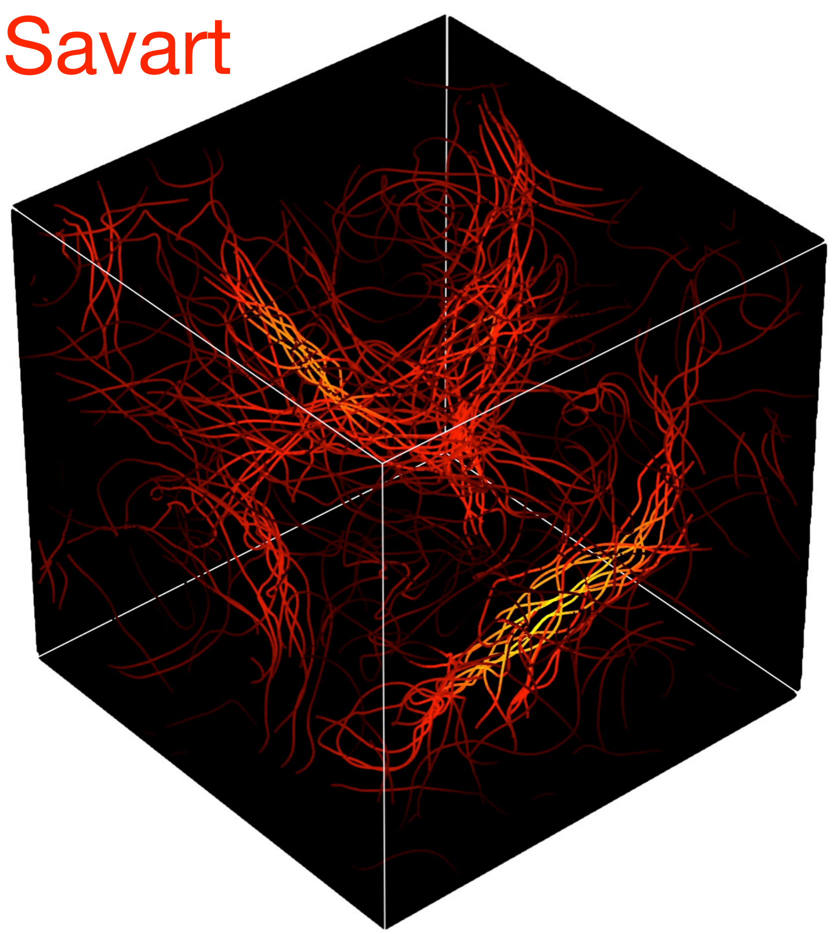

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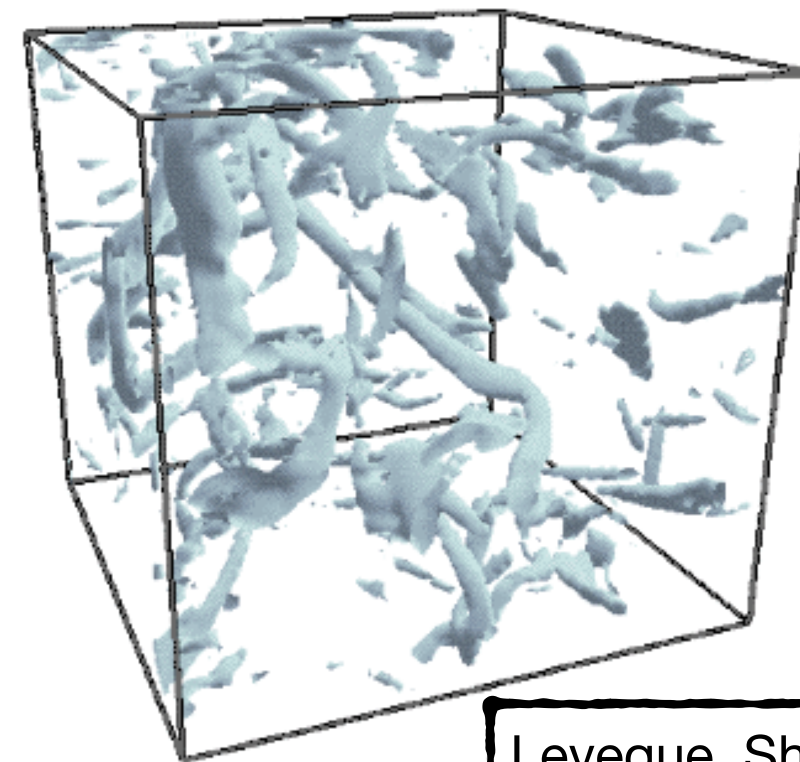

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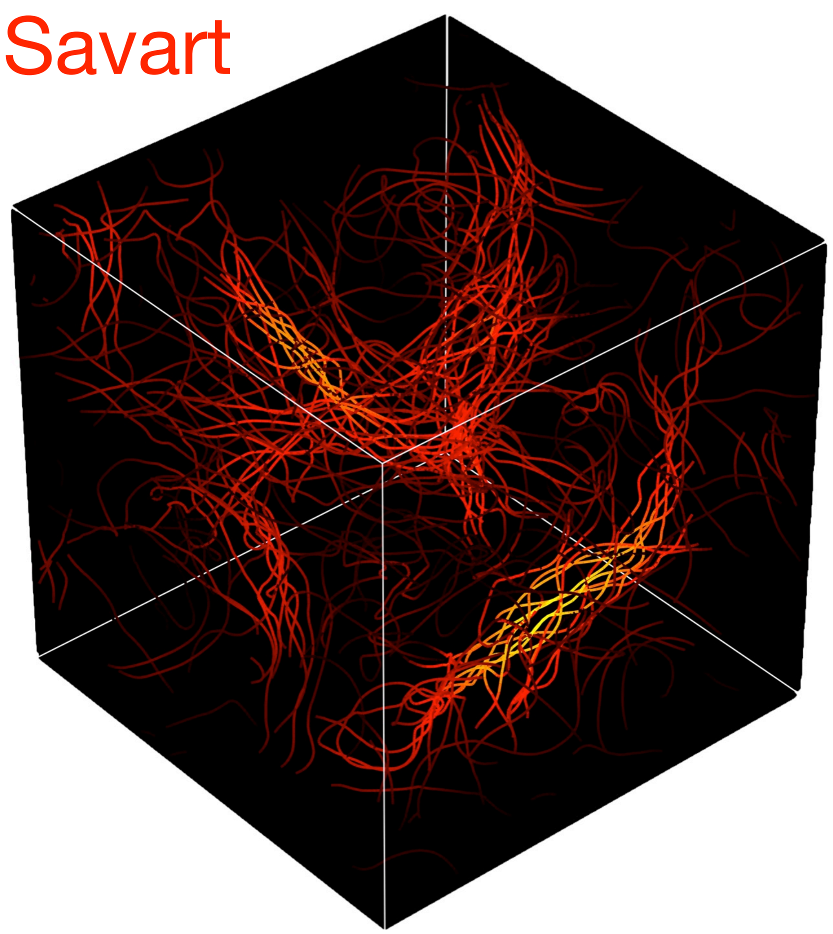

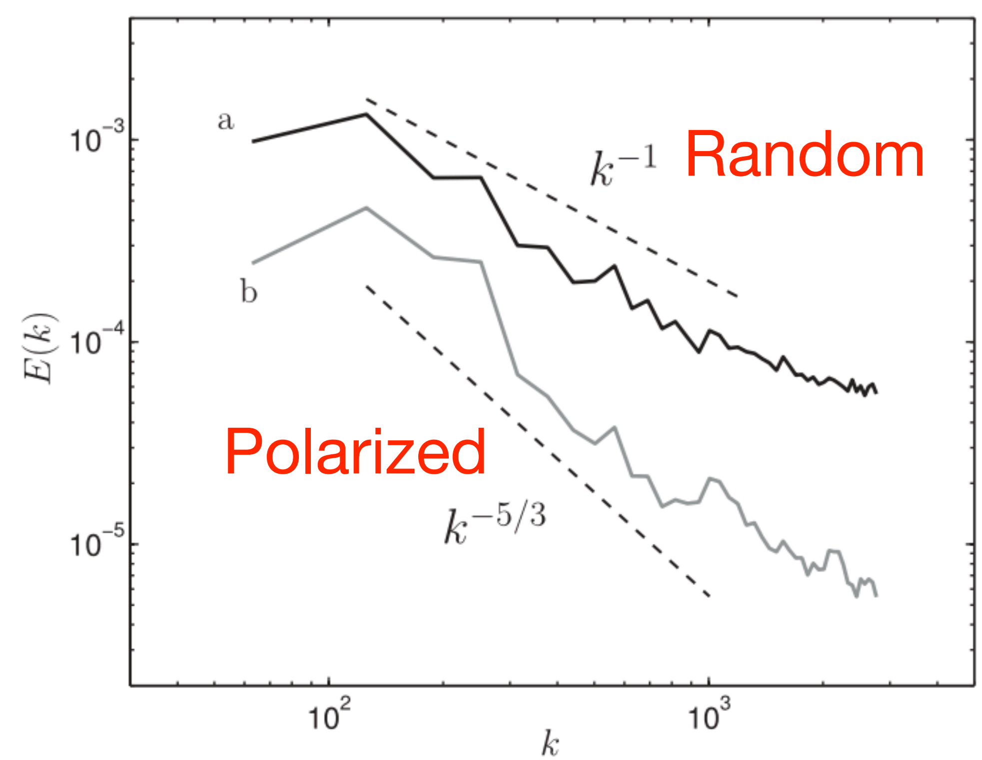

Baggaley, JL, Barenghi, Phys. Rev. Lett. **109**, 205304, (2012)

Turbulence at the Intervortex Scale

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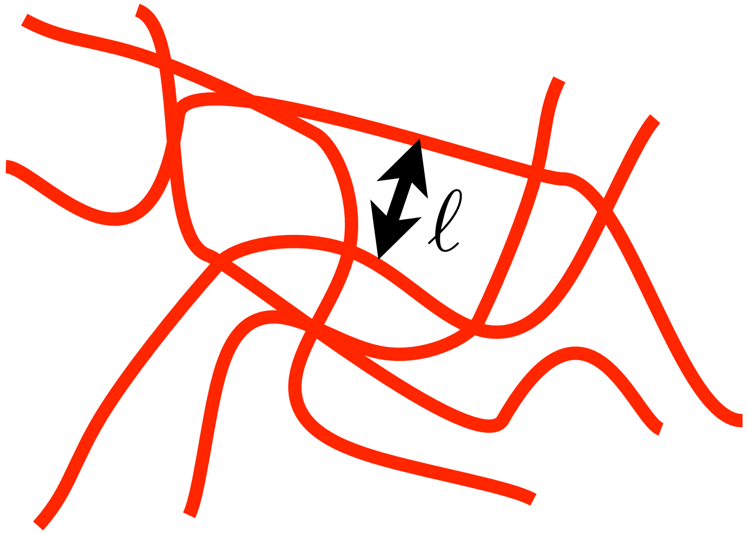
Quantum vortex reconnections

- The classical-quantum vortex bundle analogy breaks down at scales near or below the inter-vortex scale ℓ
- Quantum vortex reconnections become important for the redistribution of energy



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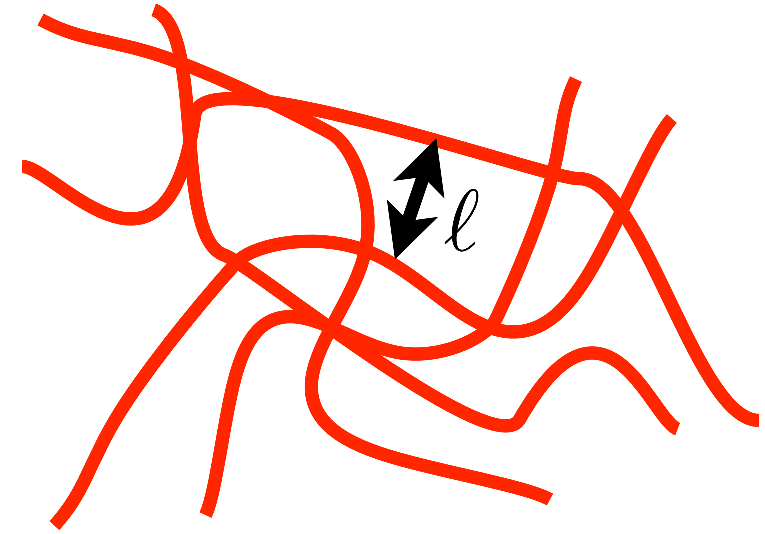

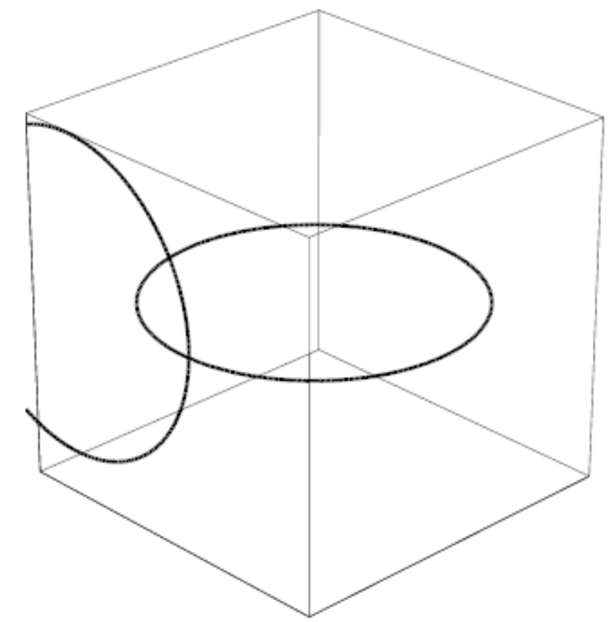

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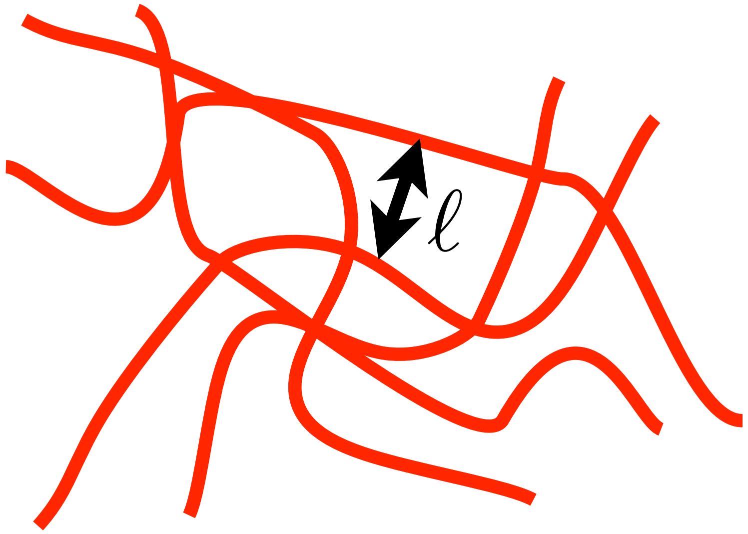

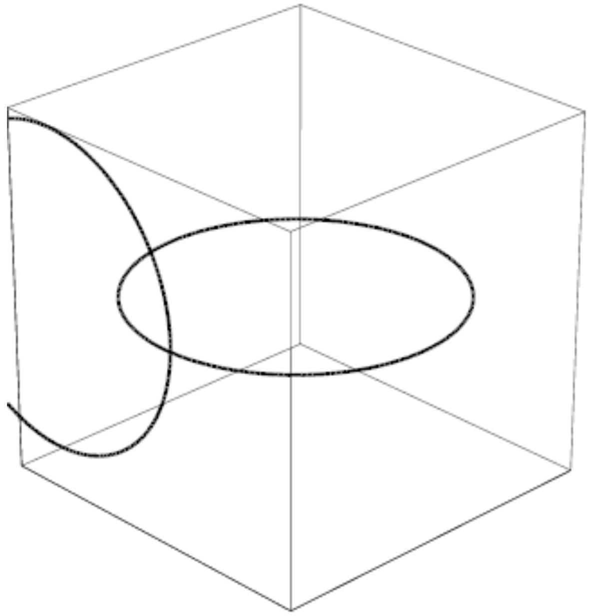

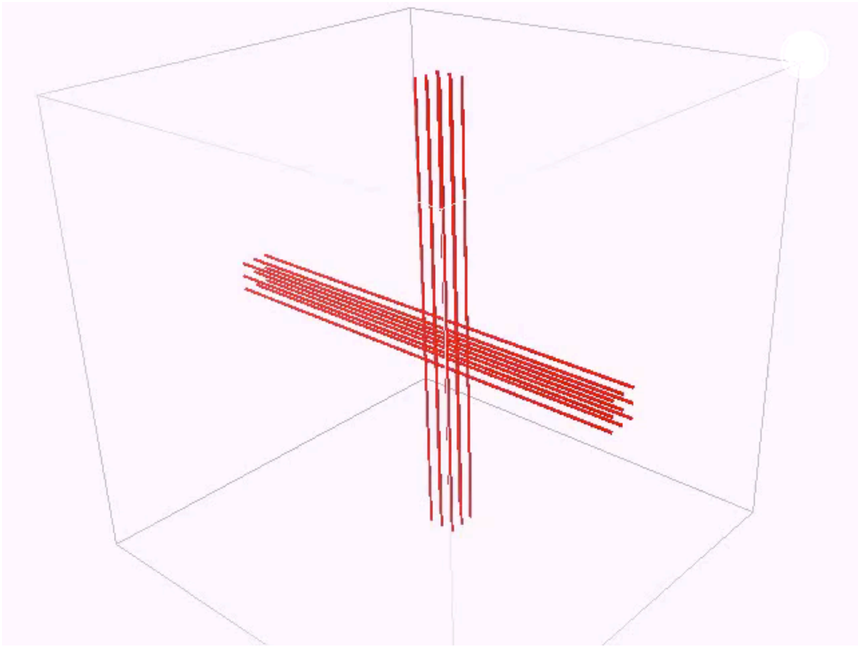

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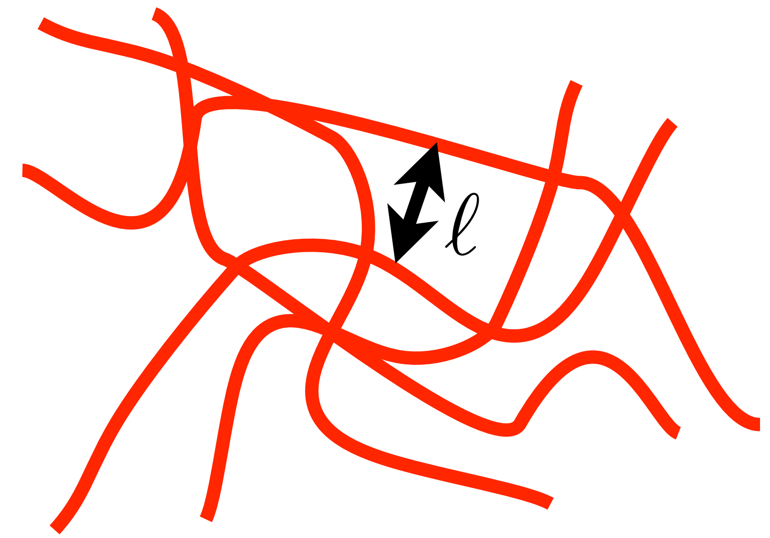

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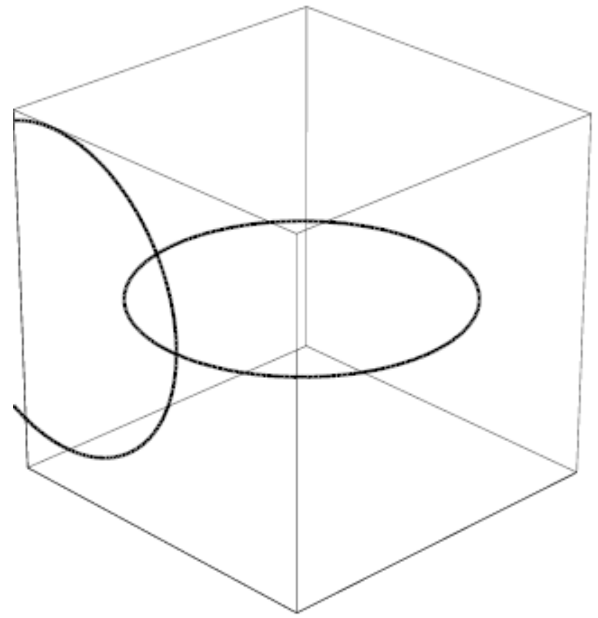
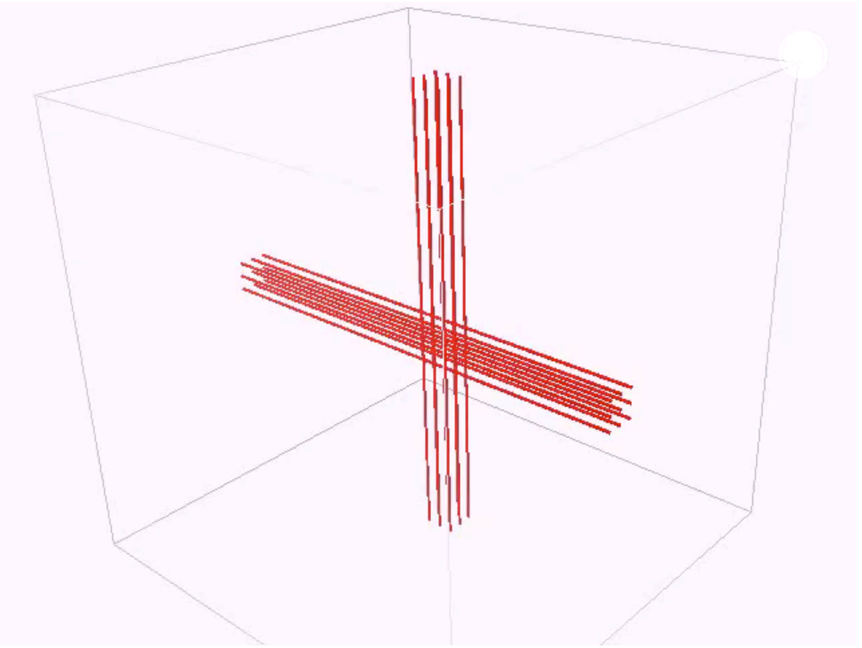


Mechanisms of energy transport

1. Vortex ring emission
    - Rings emitted from reconnection region, directly transferring energy through tangle
2. Direct sound emission
    - Phonon emission at reconnection point
3. Generation of Kelvin waves
    - Energy and momentum transferred to helical Kelvin waves that propagate along individual quantized vortex lines



Quantum Vortex Ring Emission

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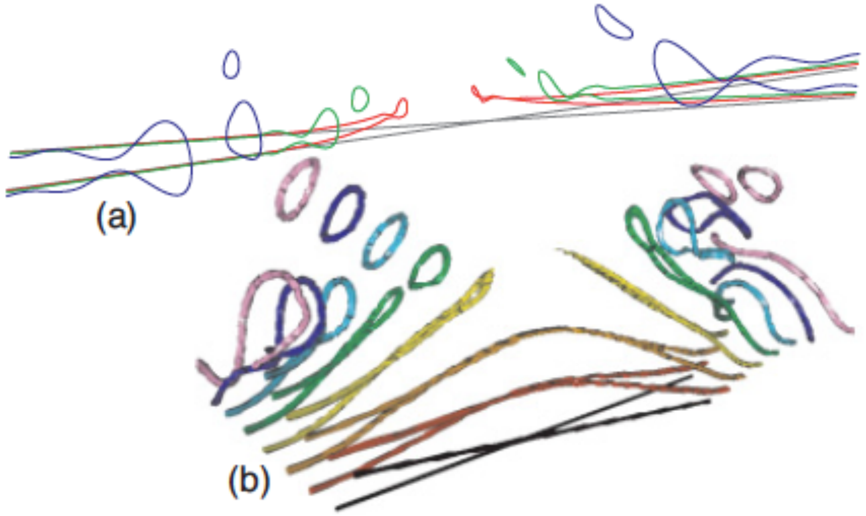
Vortex ring cascade at large angles

- A vortex reconnection of two (almost) anti-parallel vortices lead to a series of self-reconnections and the emission of multiple vortex rings
- Critical angle for ring generation in the Biot-Savart model is $\theta_c \simeq 0.942\pi$



Kursa *et al.* Phys. Rev. B, **83**, 014515, (2011)
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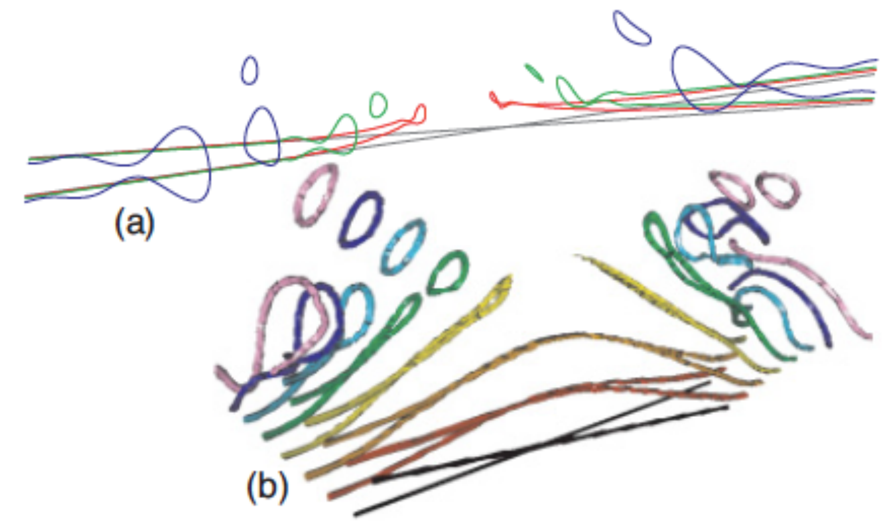

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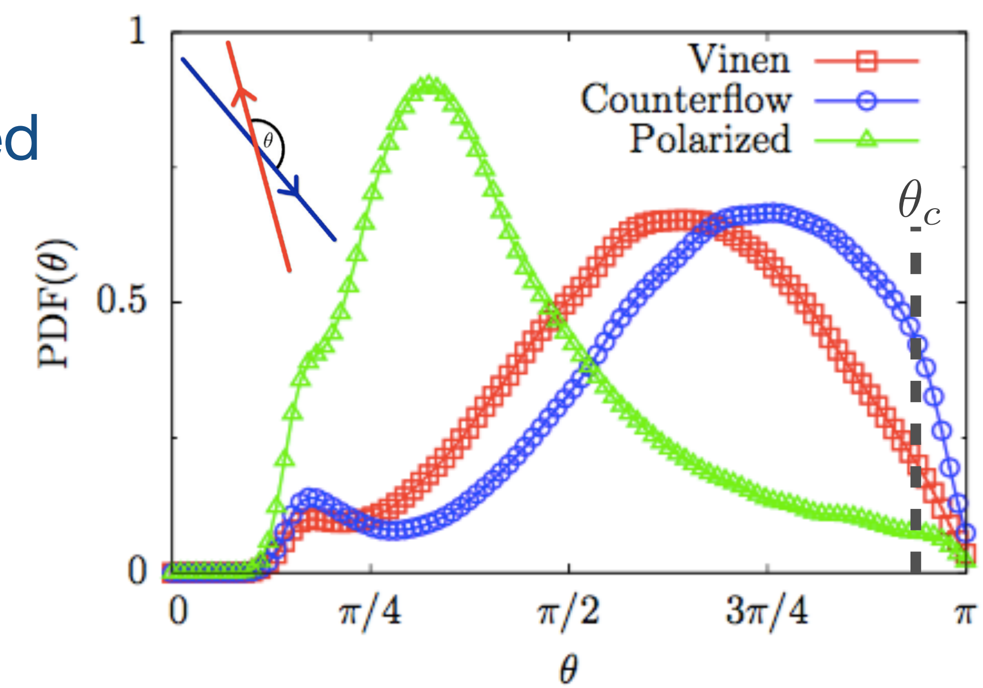
Reconnection angles in QT tangles

- Suppression of large angle reconnections in polarized tangles
- Majority of reconnections will not lead to cascade



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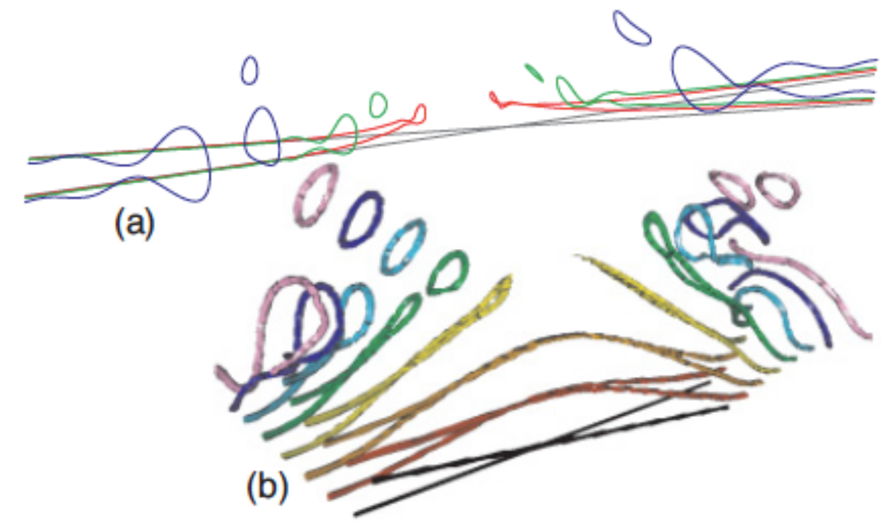

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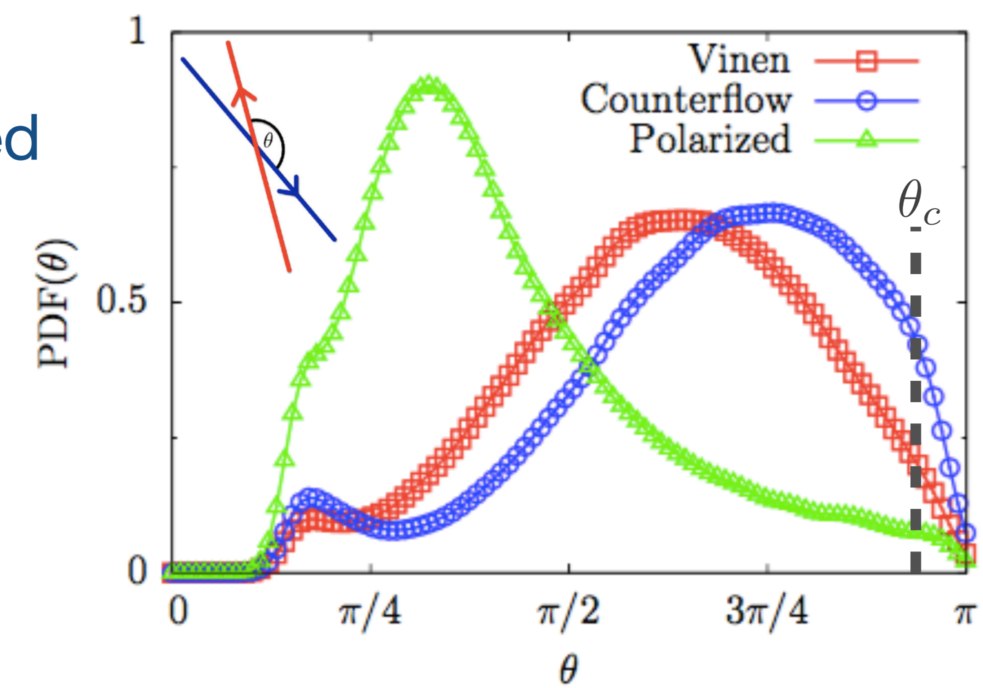

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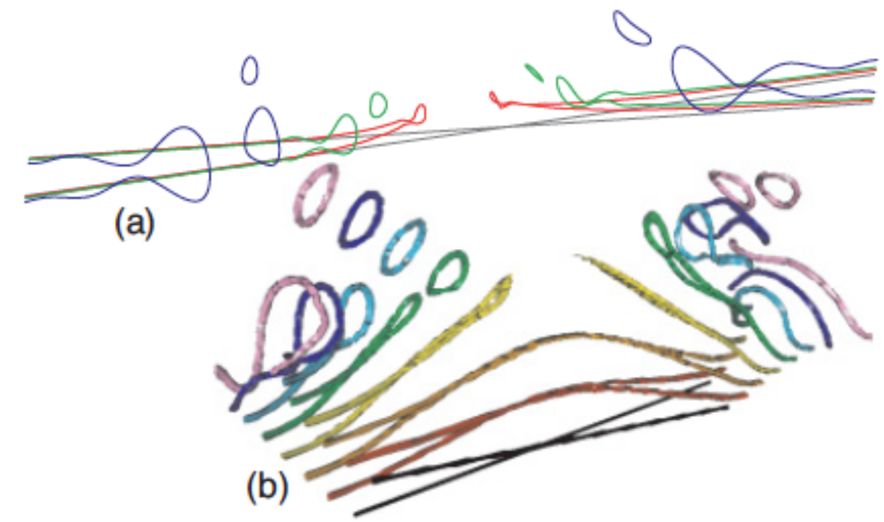

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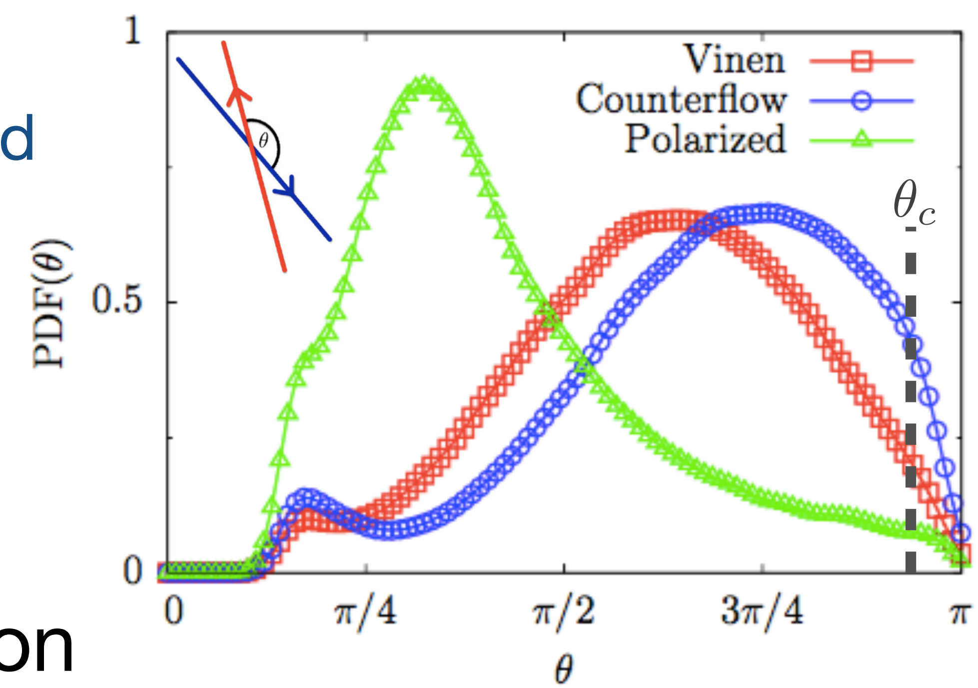
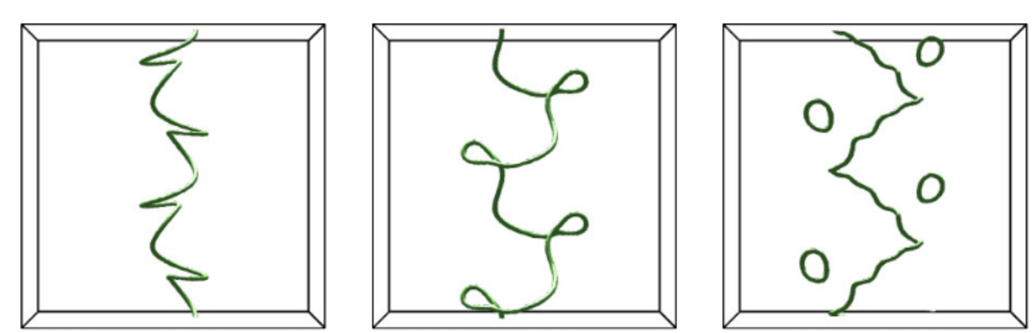


Modulational instability and self-reconnection

- Strongly nonlinear Kelvin waves can lead to modulational instability and self reconnections

Salman, Phys. Rev. Lett. **111**, 165301, (2013)



The Kelvin Wave Cascade

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Isotropic homogeneous small-scale QT

- Polarization inhibits ring emission
- Vortex reconnections transfer large-scale energy to Kelvin waves at superfluid cross-over region
- Possible *thermalisation* at the intervortex scale
- *Weakly nonlinear* Kelvin wave interactions transfer energy to even smaller scales

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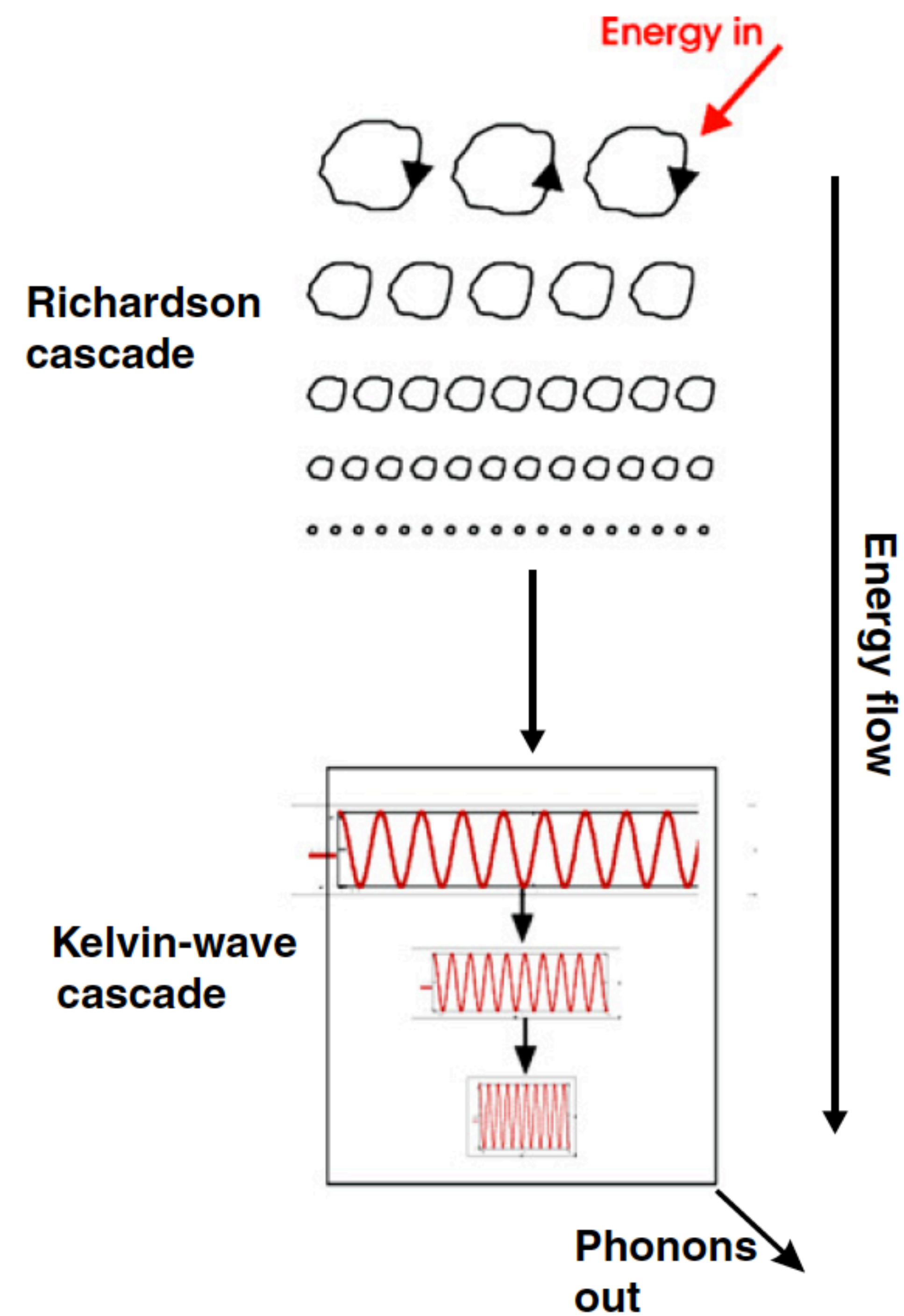

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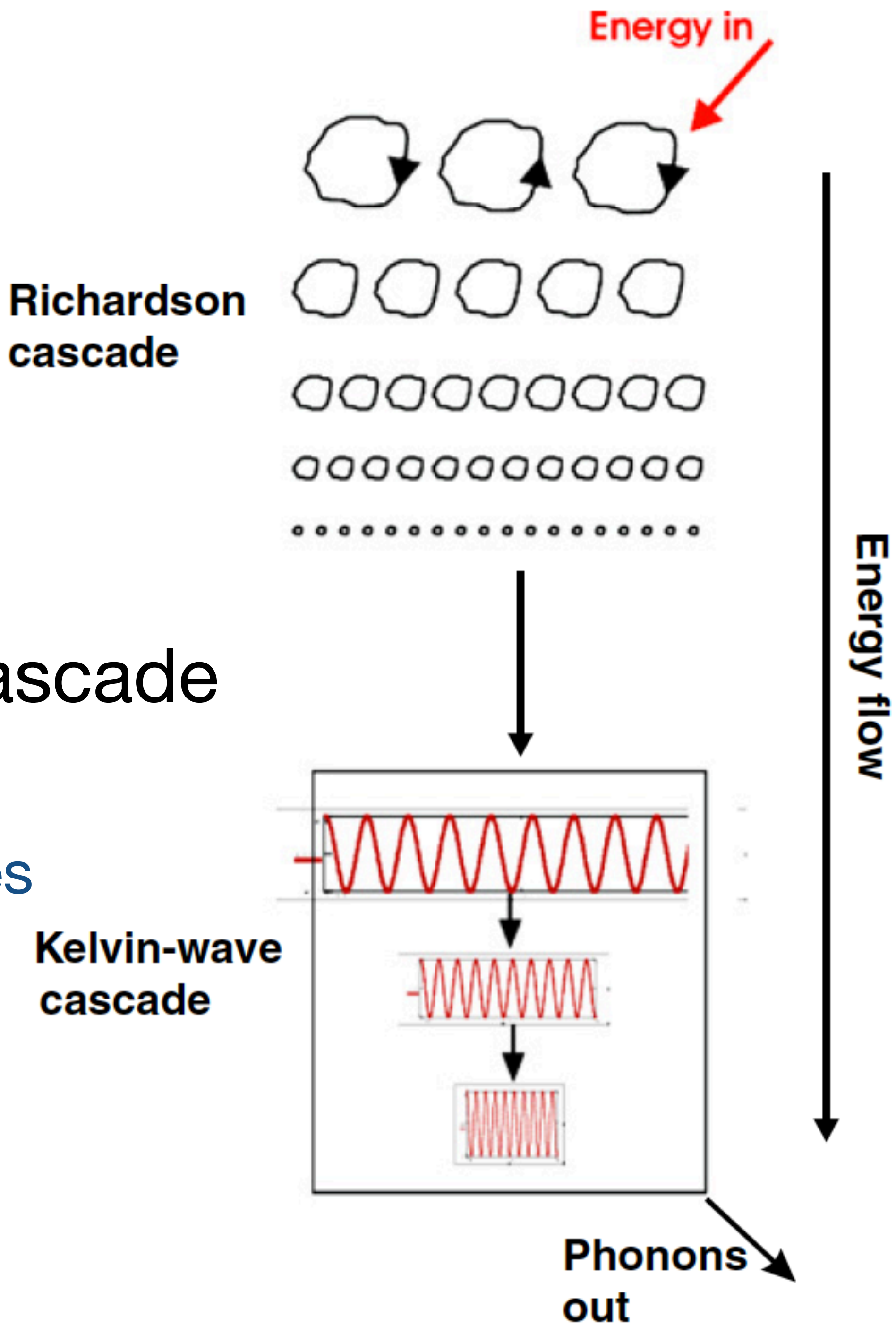
Wave turbulence description of Kelvin-wave cascade

- Theory for the non-equilibrium statistical description of the weakly nonlinear interaction of an ensemble of waves



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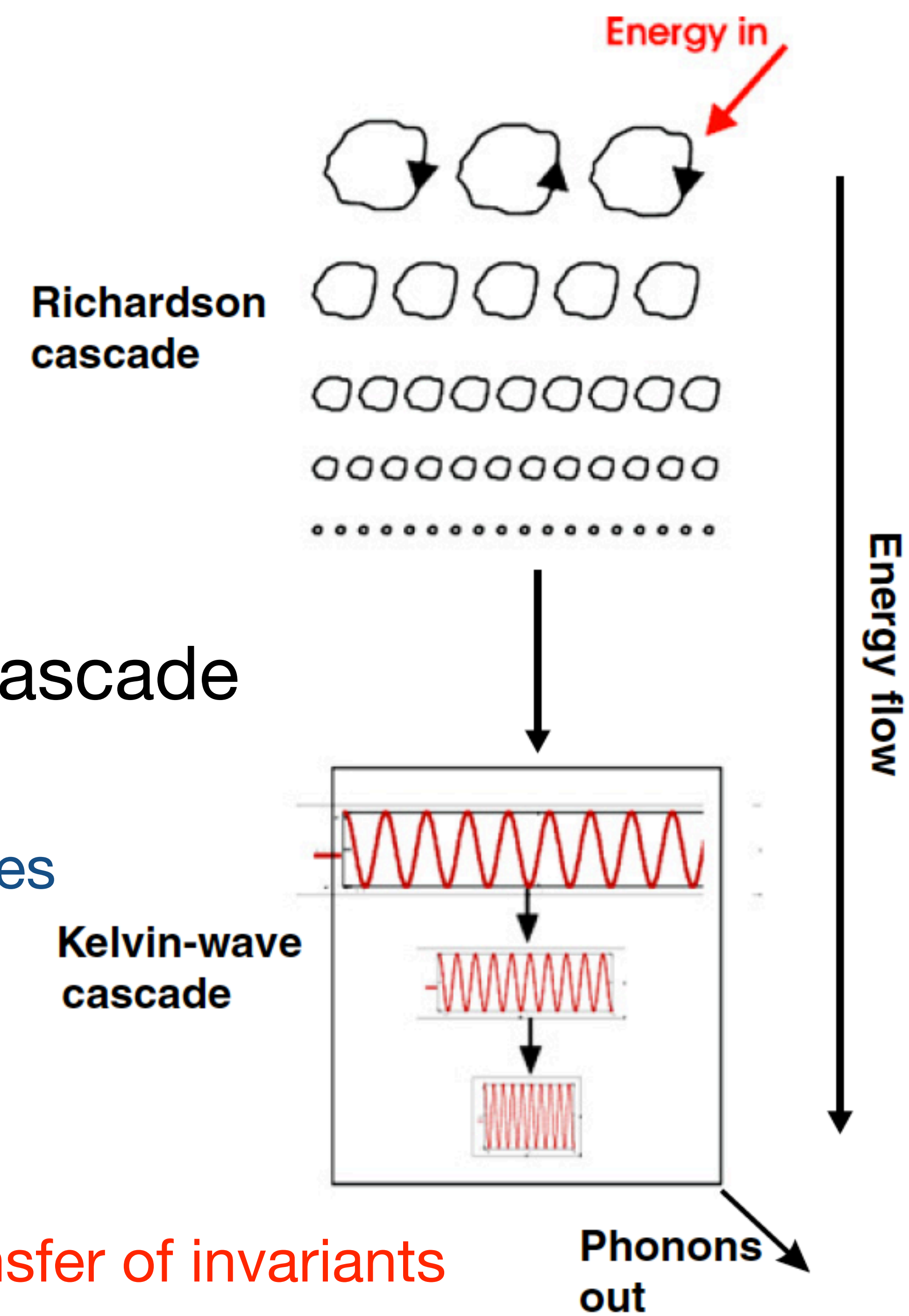

Wave turbulence description of Kelvin-wave cascade

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Main theoretical results

1. Nonlinear kinetic wave equation
2. Steady-state power-law spectra for constant flux transfer of invariants
3. But can easily study nonlinear evolution of higher-order moments and amplitude PDFs

The Wave Turbulence Setup

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Biot-Savart Hamiltonian description

$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{\mathbf{r} - \mathbf{s}}{|\mathbf{r} - \mathbf{s}|^3} \times \mathrm{d}\mathbf{r}$$

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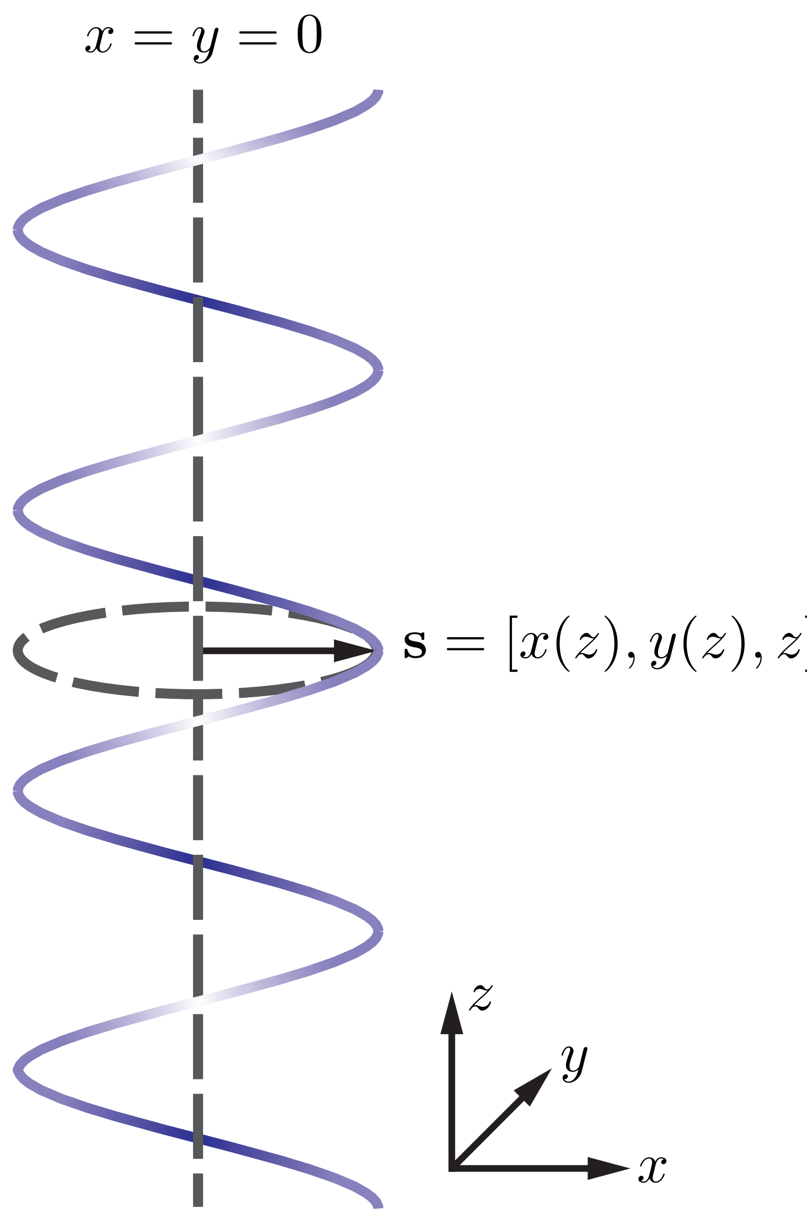

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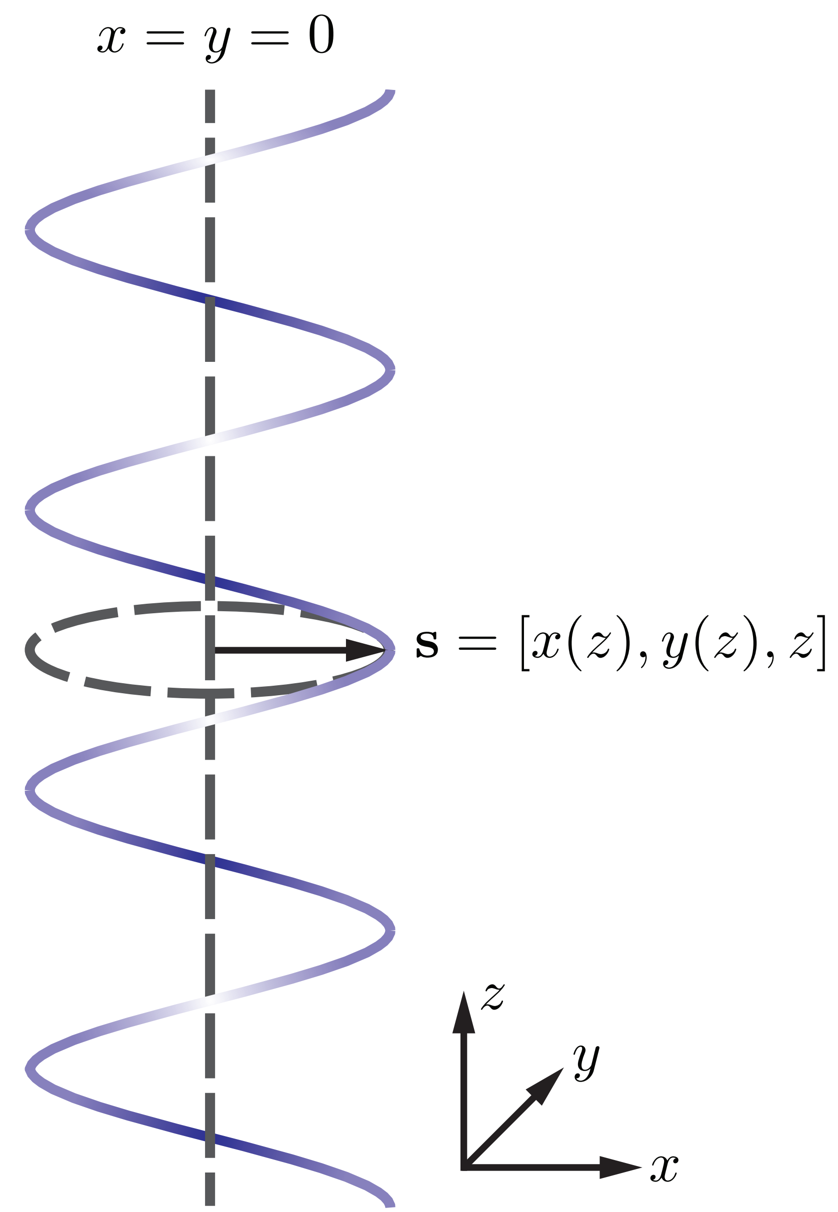

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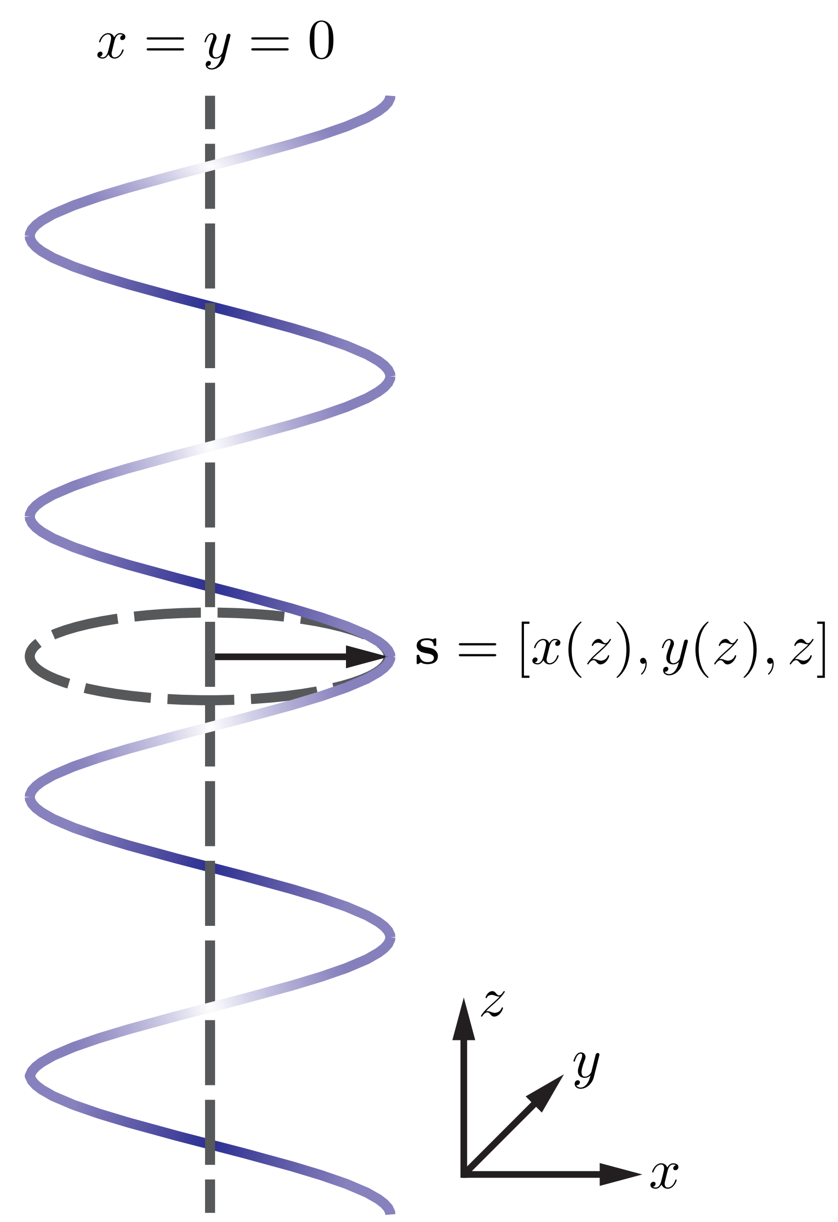

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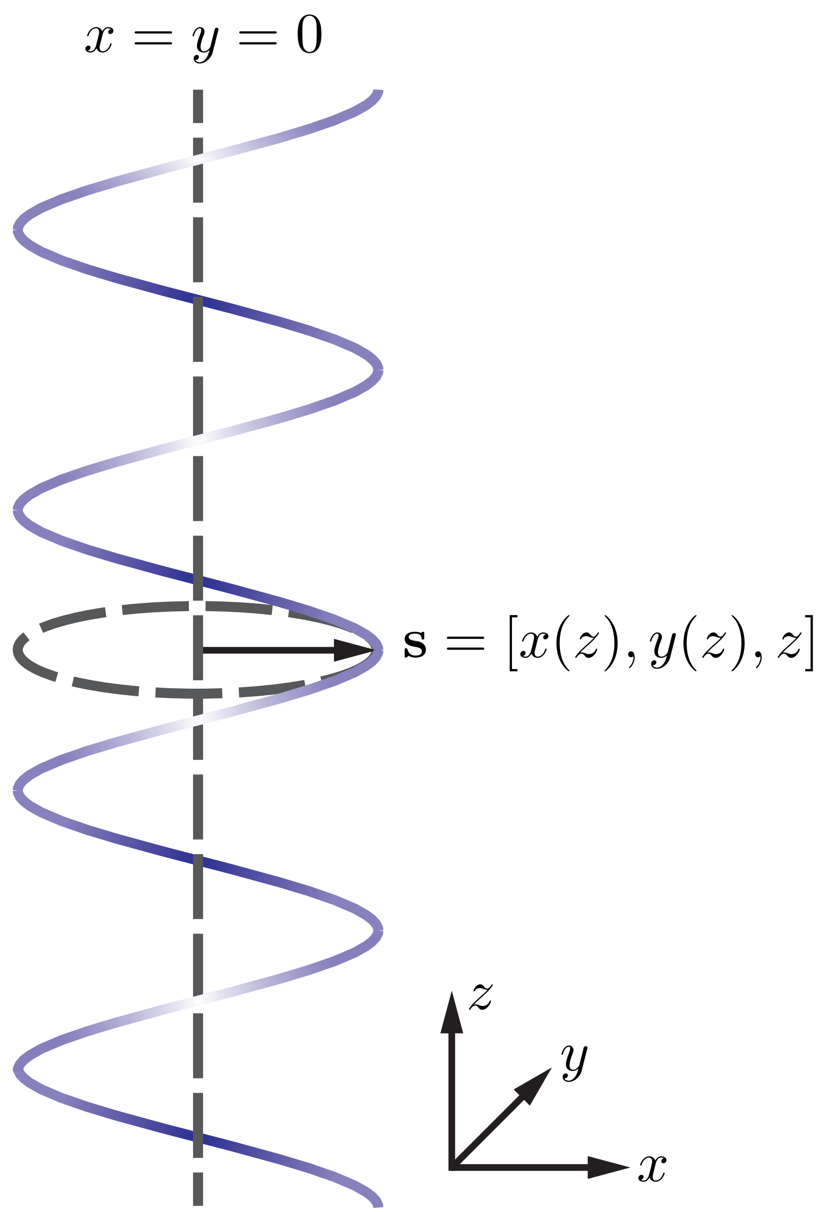
Svistunov, Phys. Rev. B, **52**, 3647, (1995)



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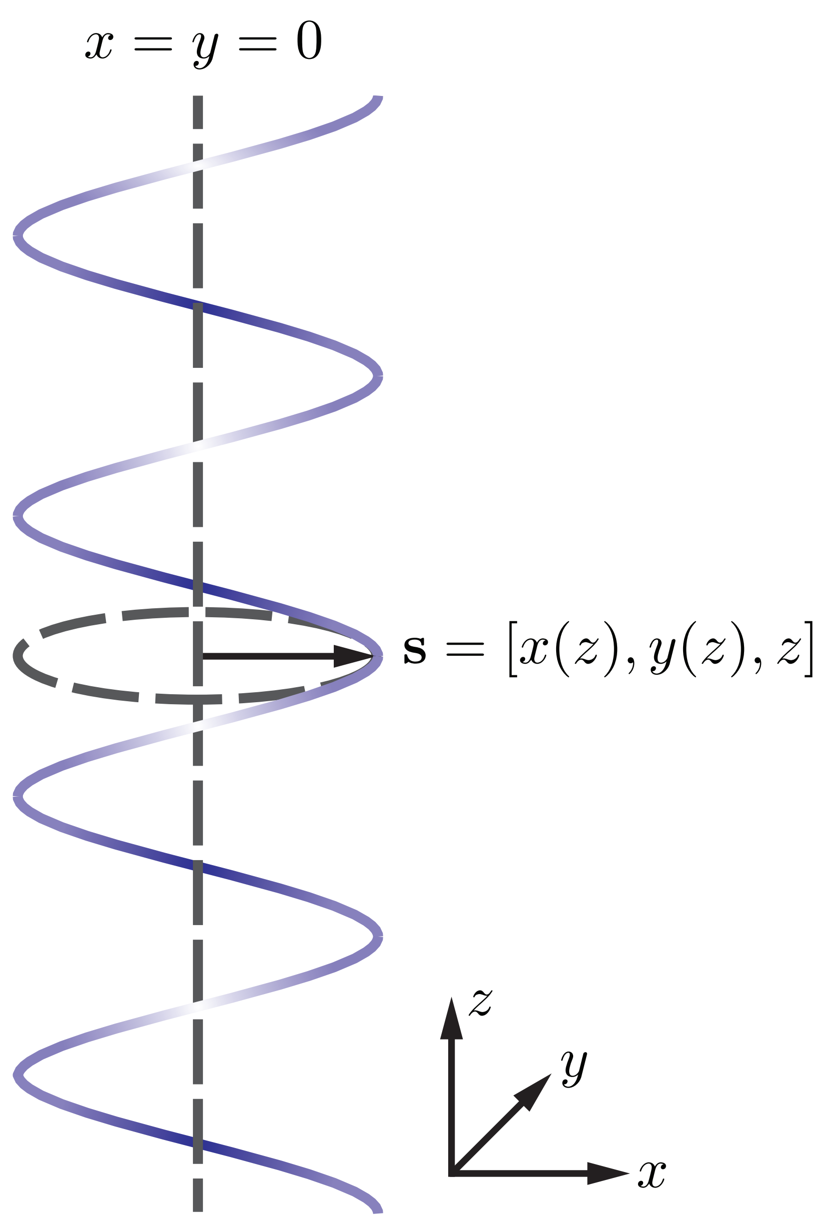

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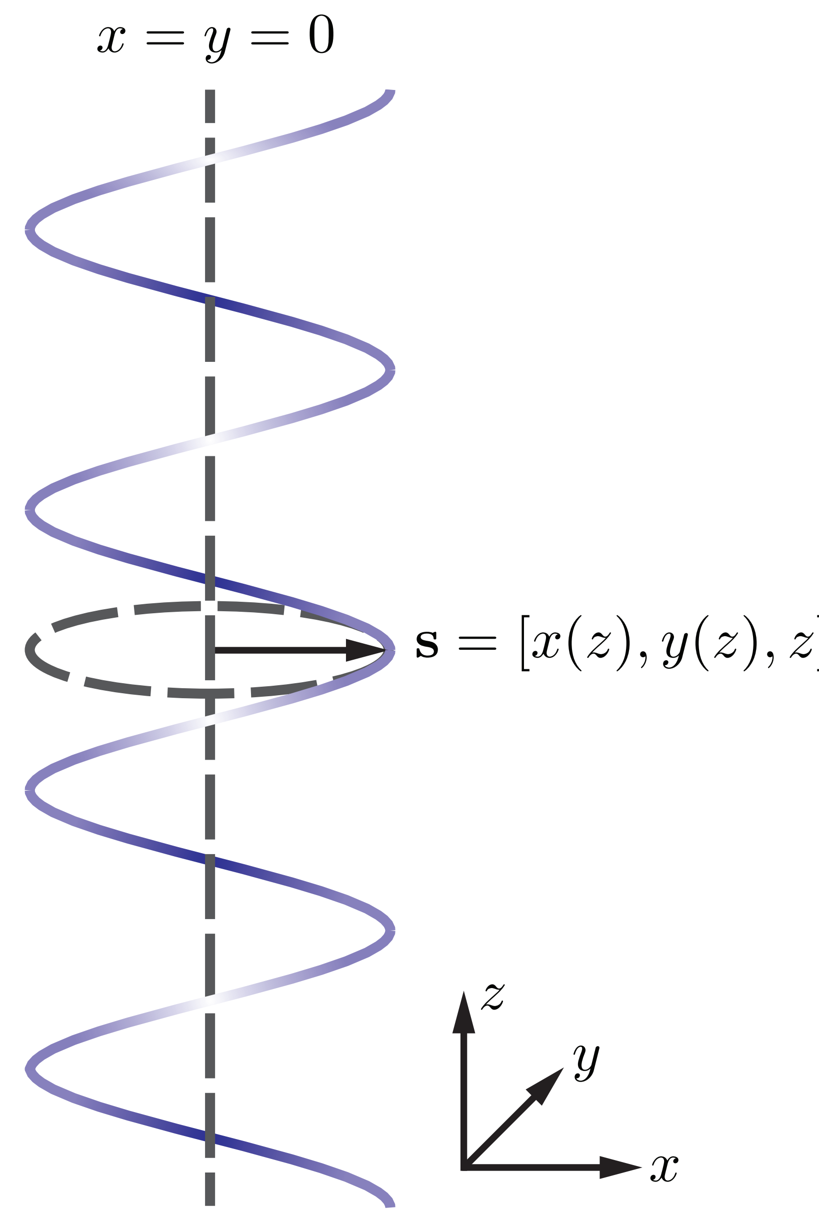

Truncation and weak nonlinear expansion

- Regularization of integral by introducing cut-off $\xi < |z_2 - z_1|$

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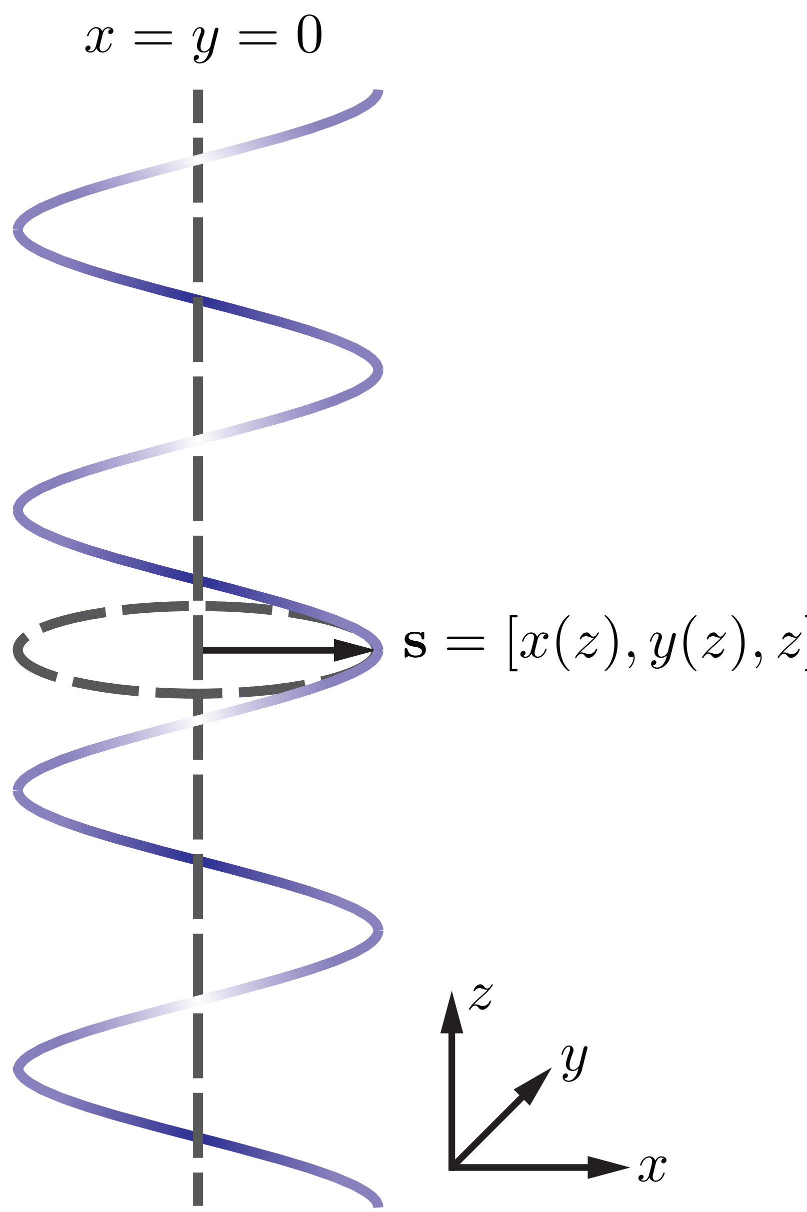

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Truncation and weak nonlinear expansion

- Regularization of integral by introducing cut-off $\xi < |z_2 - z_1|$
- Expand Hamiltonian in powers of the canonical variable:

$$\epsilon = \frac{|a(z_1) - a(z_2)|}{|z_1 - z_2|} \ll 1 \qquad \mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 + \cdots$$

Hamiltonian-Fourier Representation

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Wave action representation of the Hamiltonian

- Introduce wave action variables $a(z,t) = \kappa^{-1/2} \sum_{\mathbf{k}} a_{\mathbf{k}}(t) \exp(i\,\mathbf{k}\,z)$

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$$\mathcal{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^{*} + \frac{1}{4} \sum_{1,2,3,4} T_{3,4}^{1,2}\; a_1 a_2 a_3^{*} a_4^{*}\, \delta_{3,4}^{1,2} + \frac{1}{36} \sum_{1,2,3,4,5,6} W_{4,5,6}^{1,2,3}\; a_1 a_2 a_3 a_4^{*} a_5^{*} a_6^{*}\, \delta_{4,5,6}^{1,2,3}$$

$$a_1 = a_{\mathbf{k}_1}(t) \qquad T_{3,4}^{1,2} = T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \qquad \delta_{3,4}^{1,2} = \delta\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4\right)$$

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Interaction coefficients

$$\omega_{\mathbf{k}} = \frac{\kappa\Lambda}{4\pi}\mathbf{k}^2 - \frac{\kappa}{4\pi}\mathbf{k}^2 \ln(\mathbf{k}\ell_{\text{eff}}), \qquad \Lambda = \ln\left(\ell_{\text{eff}}/\tilde{\xi}\right) \gg 1, \quad \tilde{\xi} = \xi e^{\gamma + \frac{3}{2}}$$

$$T^{1,2}_{3,4} = -\frac{\Lambda}{4\pi}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4 - \frac{1}{16\pi}\left[5\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4 + \mathcal{F}^{1,2}_{3,4}\right]$$

$$W^{1,2,3}_{4,5,6} = \frac{9\Lambda}{8\pi\kappa}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6 + \frac{9}{32\pi\kappa}\left[7\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6 + \mathcal{G}^{1,2,3}_{4,5,6}\right]$$

- Separate logarithm divergent terms by introducing an effective length scale ℓ_{eff}
- $\mathcal{F}^{1,2}_{3,4}$ and $\mathcal{G}^{1,2,3}_{4,5,6}$ are terms containing logarithmic contributions

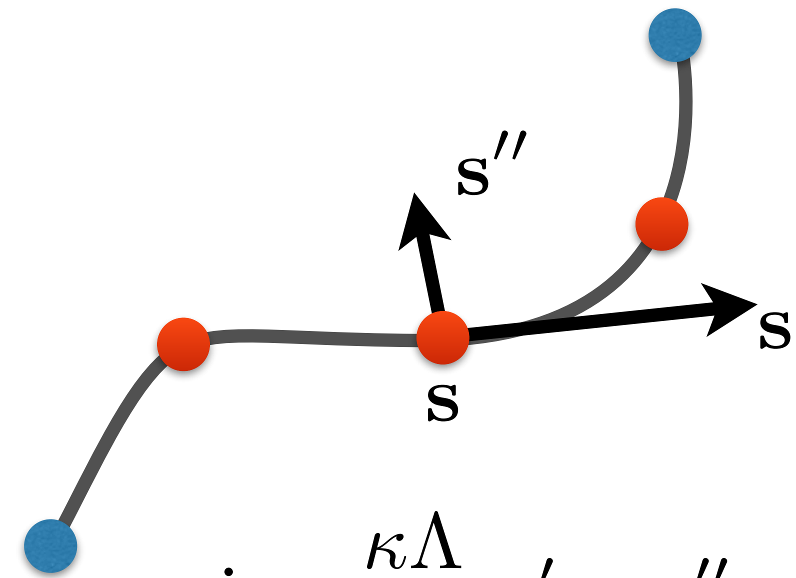
JL *et al.* Phys. Rev. B, **81**, 104526, (2010)

Leading Order Integrability

Local Induction Approximation (LIA)

- If the cutoff is small then terms proportional to Λ give greatest contribution and diverge in the limit $\xi \to 0$
- Keeping only the leading divergent terms, then the Hamiltonian becomes

$$\mathcal{H} = \frac{\kappa^2 \Lambda}{2\pi} \int \sqrt{1 + |a'(z)|^2}\, \mathrm{d}z$$



$$\dot{\mathbf{s}} = \frac{\kappa \Lambda}{4\pi} \mathbf{s}' \times \mathbf{s}''$$

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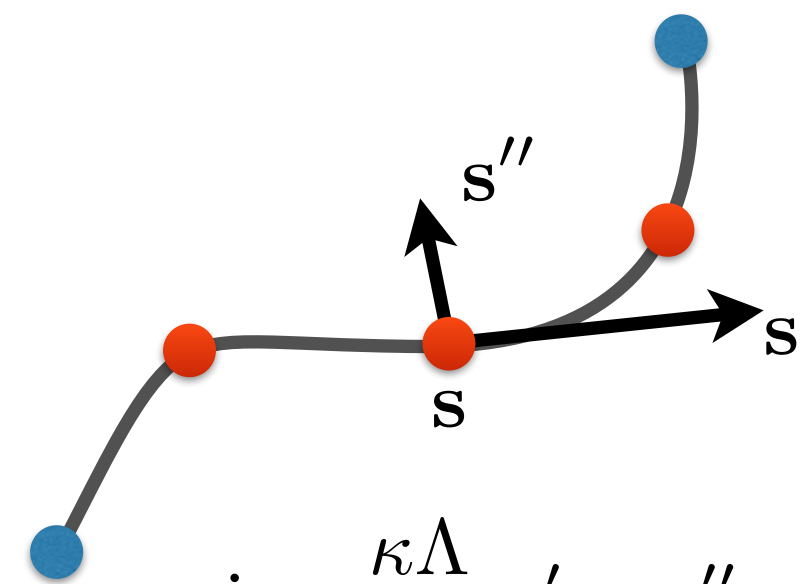

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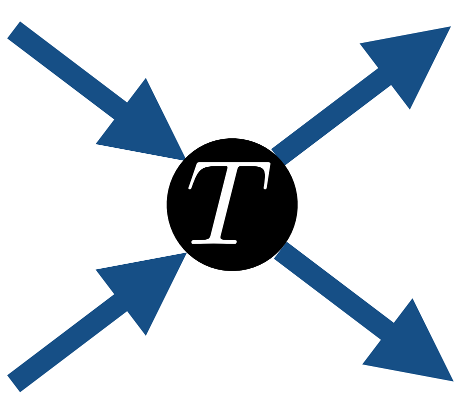

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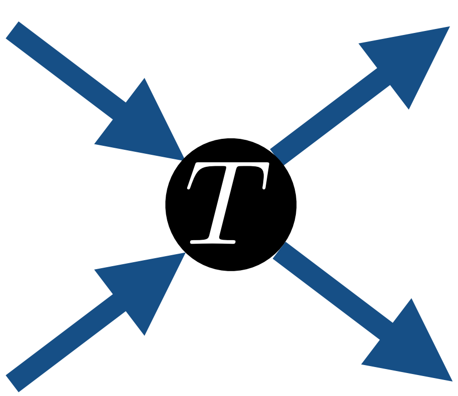

- Only trivial resonances can solve resonance condition for Kelvin-wave frequency

$$\mathbf{k}_1 = \mathbf{k}_3, \quad \mathbf{k}_2 = \mathbf{k}, \quad \text{or} \quad \mathbf{k}_1 = \mathbf{k}, \quad \mathbf{k}_2 = \mathbf{k}_3$$

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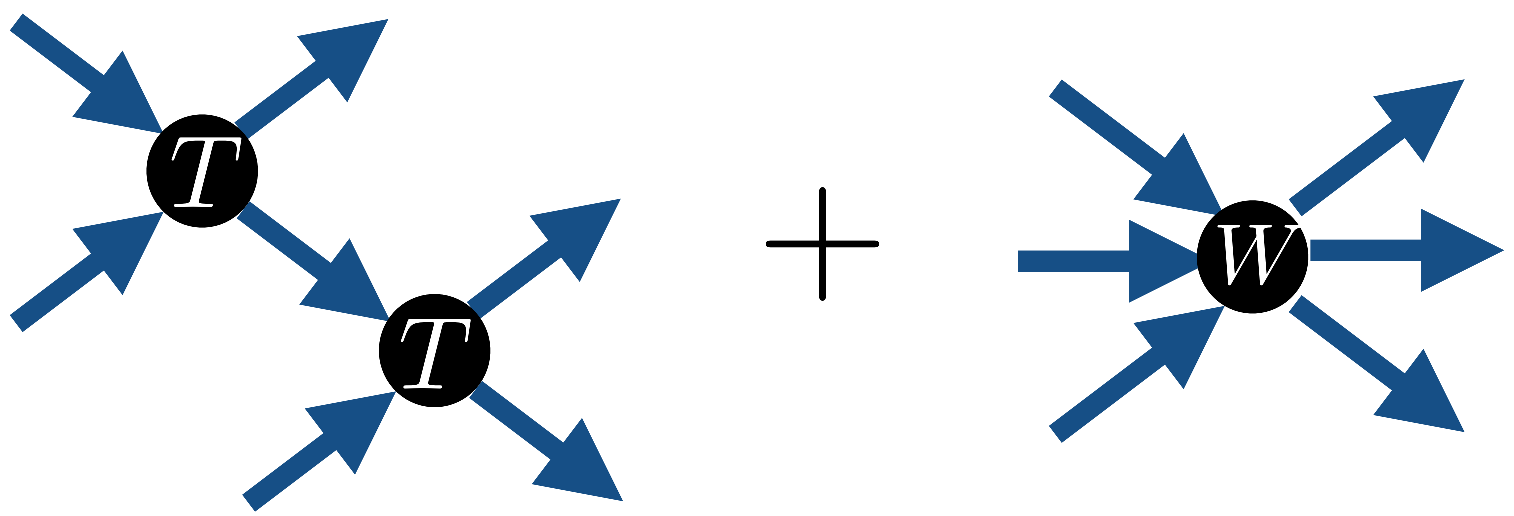

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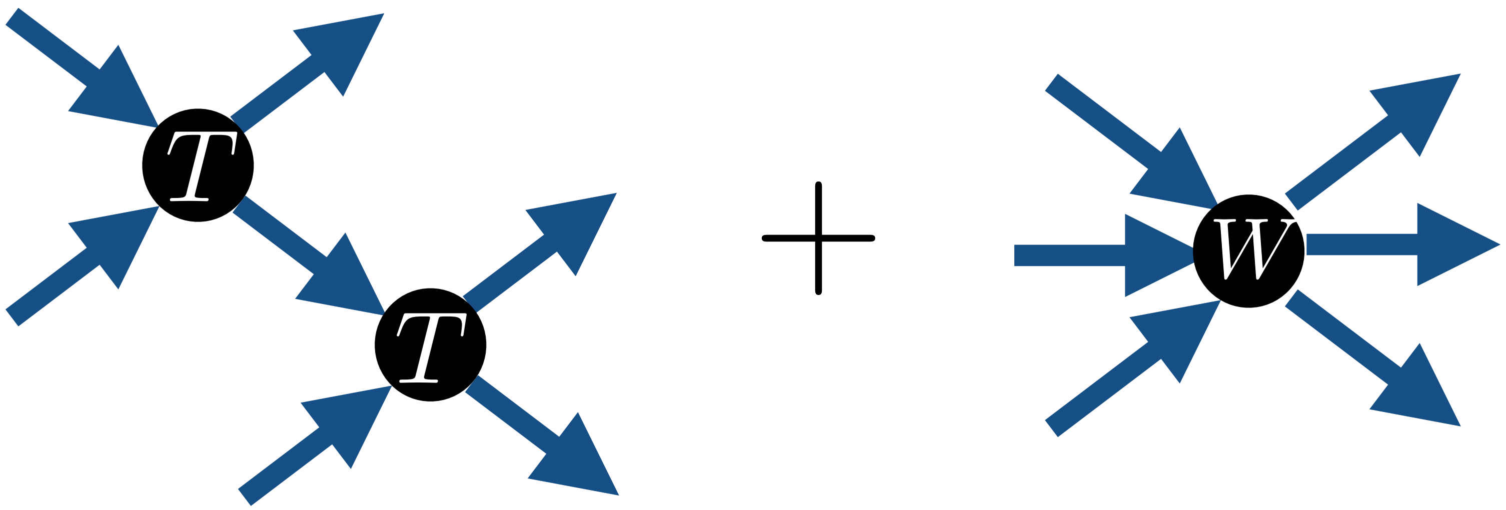

Six-wave interaction coefficient of $\mathcal{H}_6$

JL *et al.* Phys. Rev. B, **81**, 104526, (2010)

$$\tilde{W} = W^{\Lambda} + \frac{T^{\Lambda} \circ T^{\Lambda}}{\omega^{\Lambda}} + W^{1} + \frac{T^{1} \circ T^{\Lambda}}{\omega^{\Lambda}} + \frac{T^{\Lambda} \circ T^{1}}{\omega^{\Lambda}} + \frac{T^{\Lambda} \circ T^{\Lambda}}{\left(\omega^{\Lambda}\right)^{2}} \omega^{1} + \mathcal{O}\left(\Lambda^{-1}\right)$$

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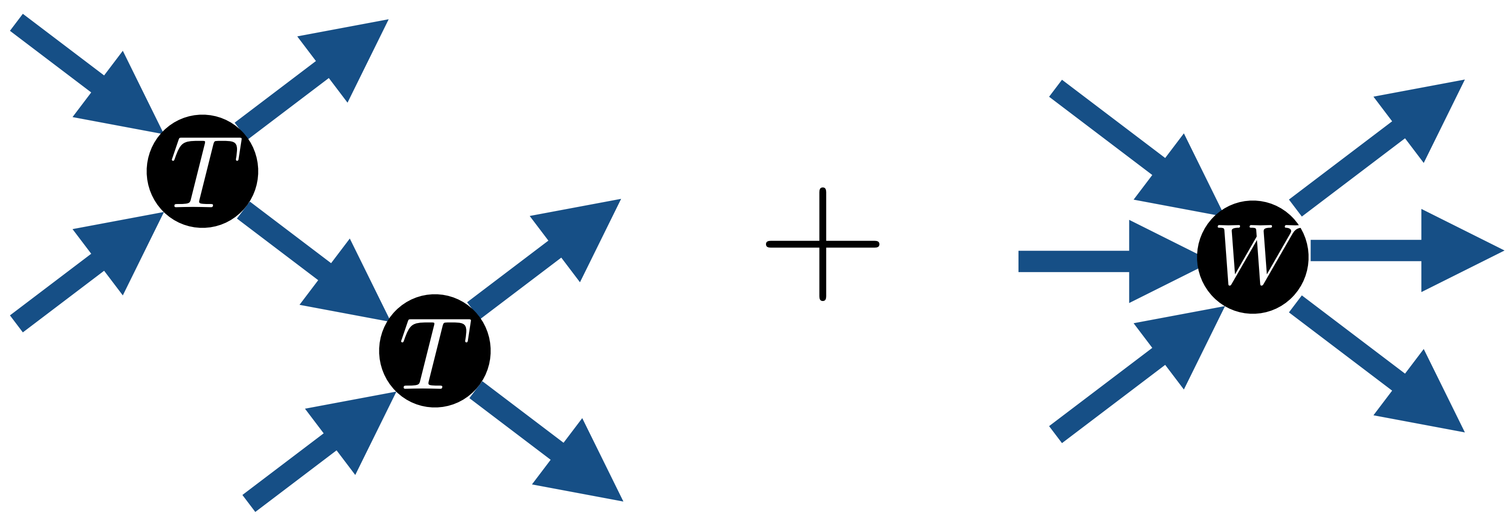

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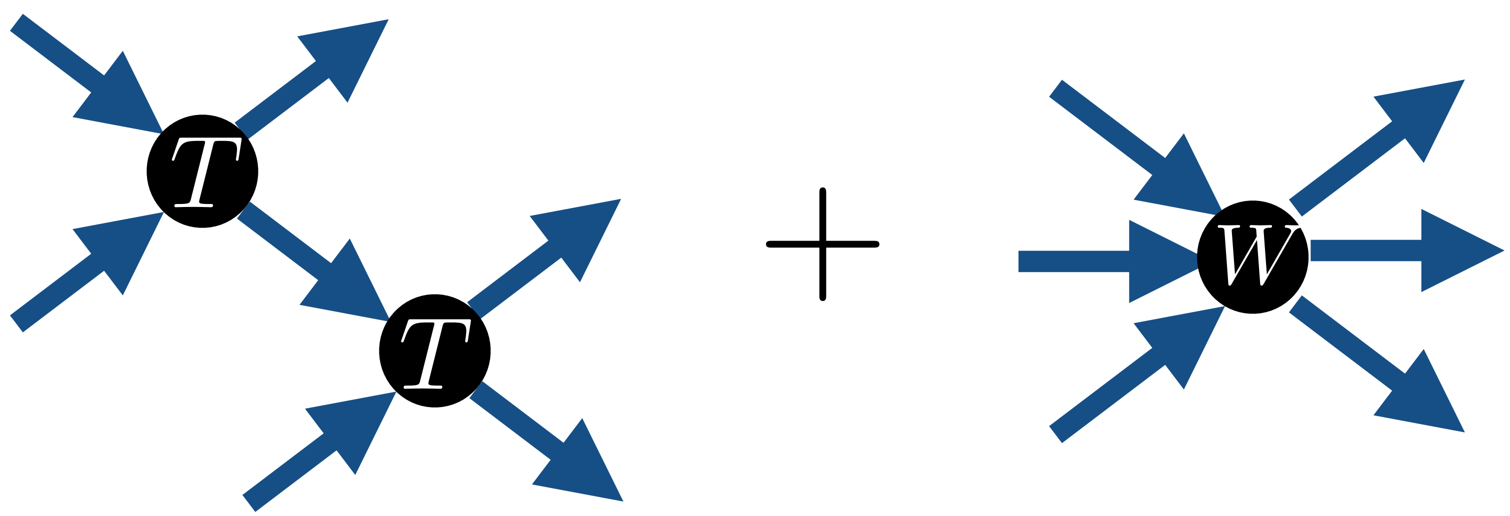

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Leading order terms describing Kelvin-wave dynamics

Kinetic Wave Equation

Wave action density

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Kozik, Svistunov, Phys. Rev. Lett., **92**, 035301, (2004)

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Kolmogorov-Zakharov power-law solutions $\frac{\partial n_{\mathbf{k}}}{\partial t} = I\left(n_{\mathbf{k}} = C k^{-x}\right) = 0$

- One solution corresponds to constant energy transfer to small scales by Kelvin-waves

$$E_k = C_{KS} \, \Lambda \, \kappa^{7/5} \, \epsilon^{1/5} \, k^{-7/5}$$

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Kolmogorov-Zakharov power-law solutions $\frac{\partial n_{\mathbf{k}}}{\partial t} = I\left(n_{\mathbf{k}} = C k^{-x}\right) = 0$

- One solution corresponds to constant energy transfer to small scales by Kelvin-waves

$$E_k = C_{KS} \, \Lambda \, \kappa^{7/5} \, \epsilon^{1/5} \, k^{-7/5}$$

Kozik-Svistunov Energy Spectrum

Kinetic Wave Equation

Aston University

Wave action density

- Of particular interest is the second order correlator function $\langle a_{\mathbf{k}} a^*_{\mathbf{k}_1} \rangle = n_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}_1)$
- *Wave* energy density is related to the wave action by $E_{\mathbf{k}} = \omega_{\mathbf{k}} n_{\mathbf{k}}$

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Kozik-Svistunov Energy Spectrum

Locality

- With any KZ solutions, convergence of the collision integral must be ensured in order for the realizability of the stationary state

Effective Four-Wave Description

Nonlocal wave interactions

- *Exact calculation* of interaction coefficient enabled us to prove that six-wave collision integral diverges in the limit of two long Kelvin-waves

Effective Four-Wave Description

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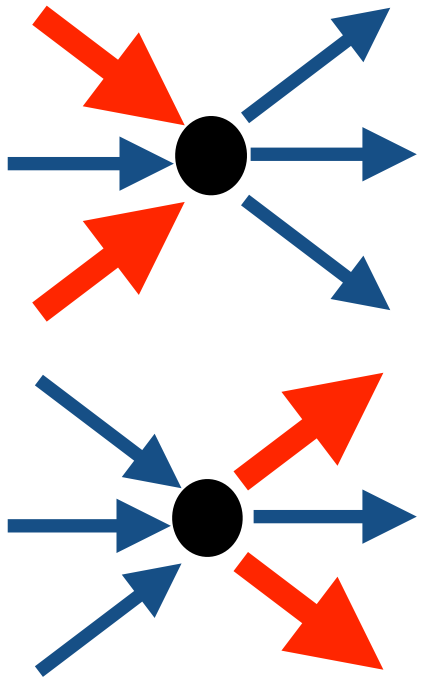

JL *et al.* Phys. Rev. B, **81**, 104526, (2010)
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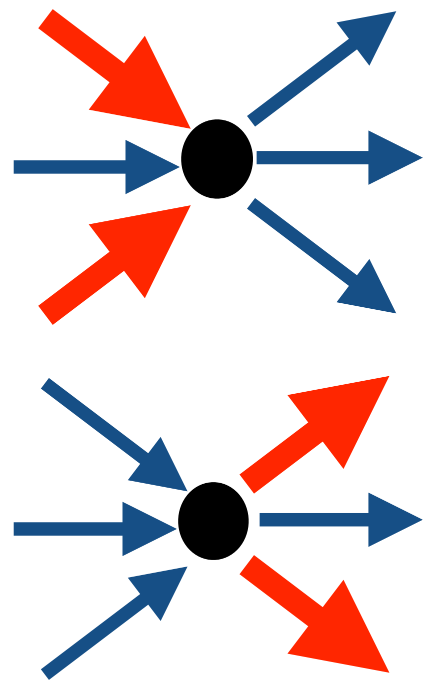

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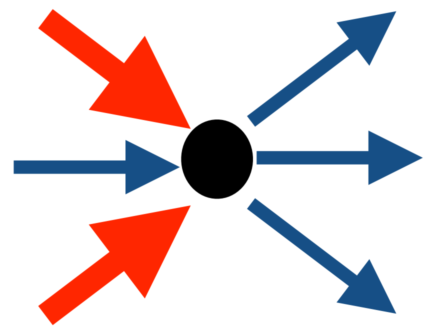

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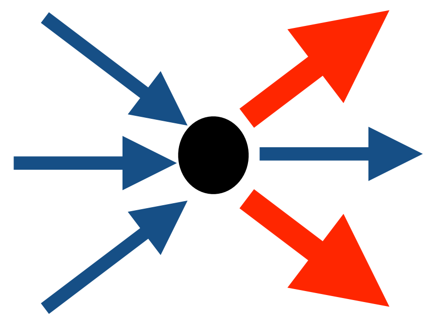

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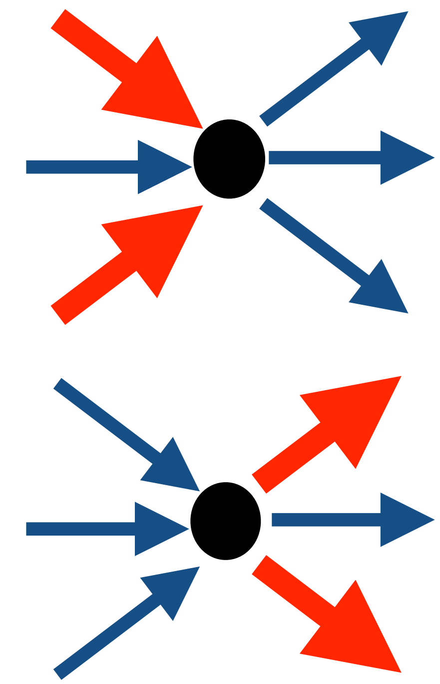

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Local vs Nonlocal Wave Interaction

Aston University

- Six-wave description with *assumed locality:* KS: $E_k = C_{KS}\,\Lambda\,\kappa^{7/5}\,\epsilon^{1/5}\,k^{-7/5}$
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Boué *et al.* Phys. Rev. B, **84**, 064516, (2011)

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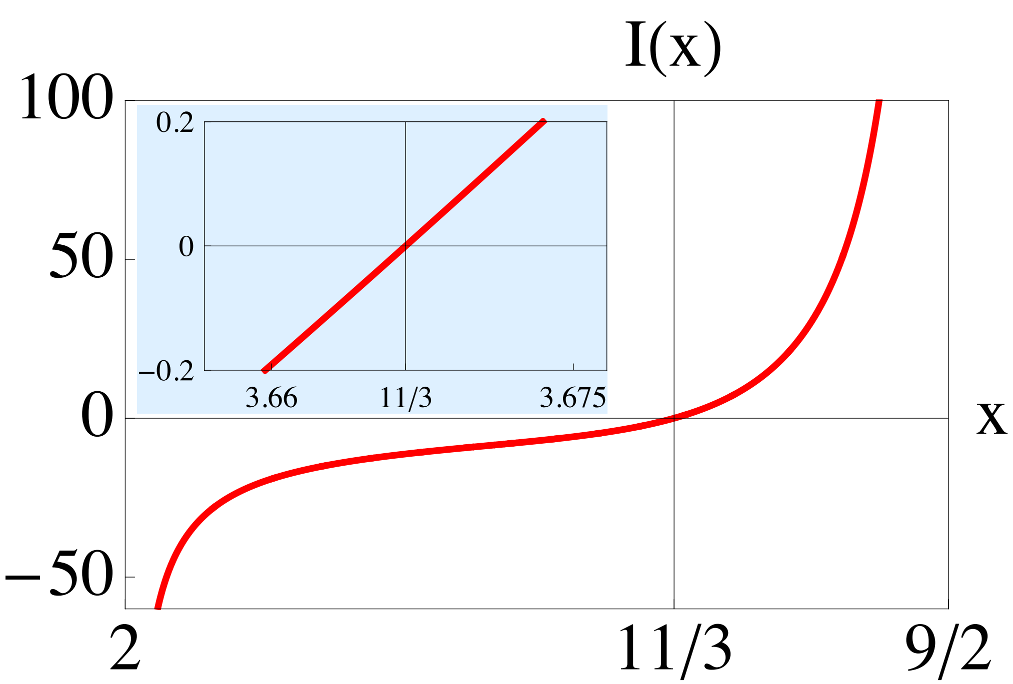

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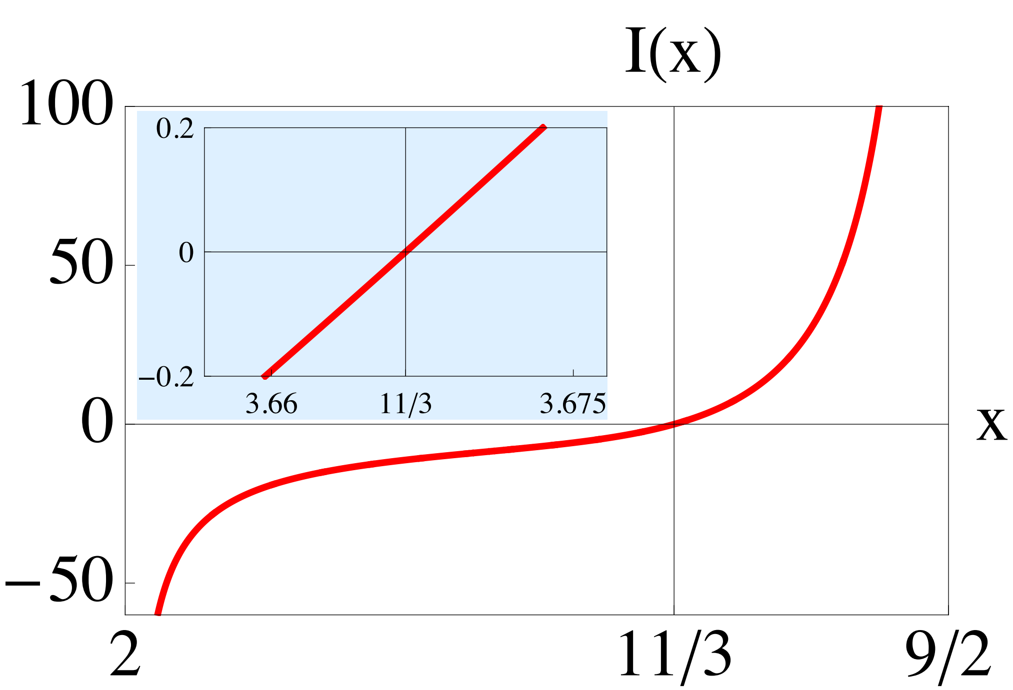

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L'vov-Nazarenko spectrum prefactor

$$C_{LN} = (128\pi)^{1/3}\left(\left.\frac{dI(x)}{dx}\right|_{x=11/3}\right)^{-1/3} = 0.304$$

Identification of Spectrum

History of simulations

- Many previous simulations (Vinen, Tsubota; Kozik, Svisuntov; Barenghi, Baggaley)
- Resolution not sufficient to distinguish between *KS* and *LN* spectrum predictions

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Aston University

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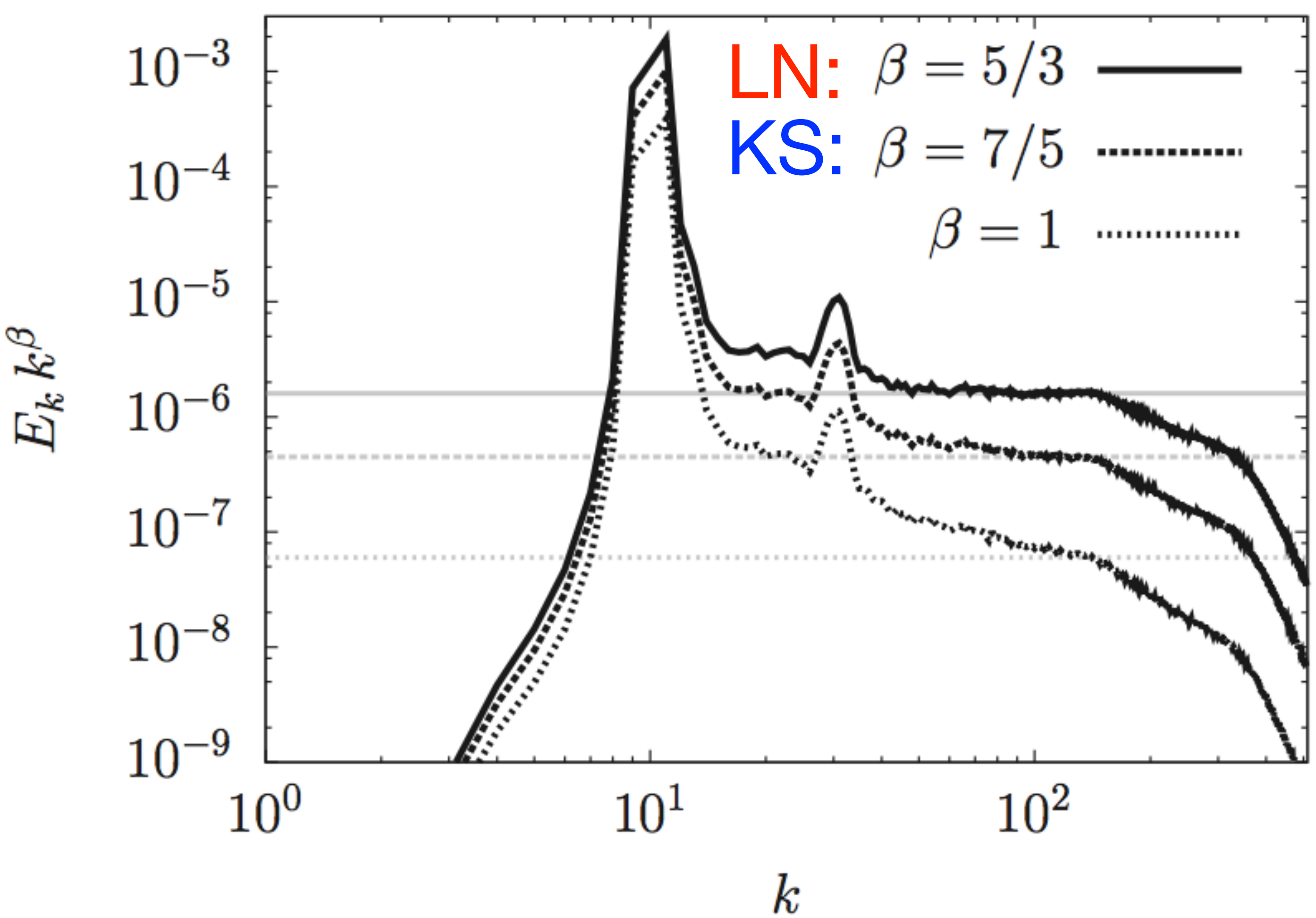

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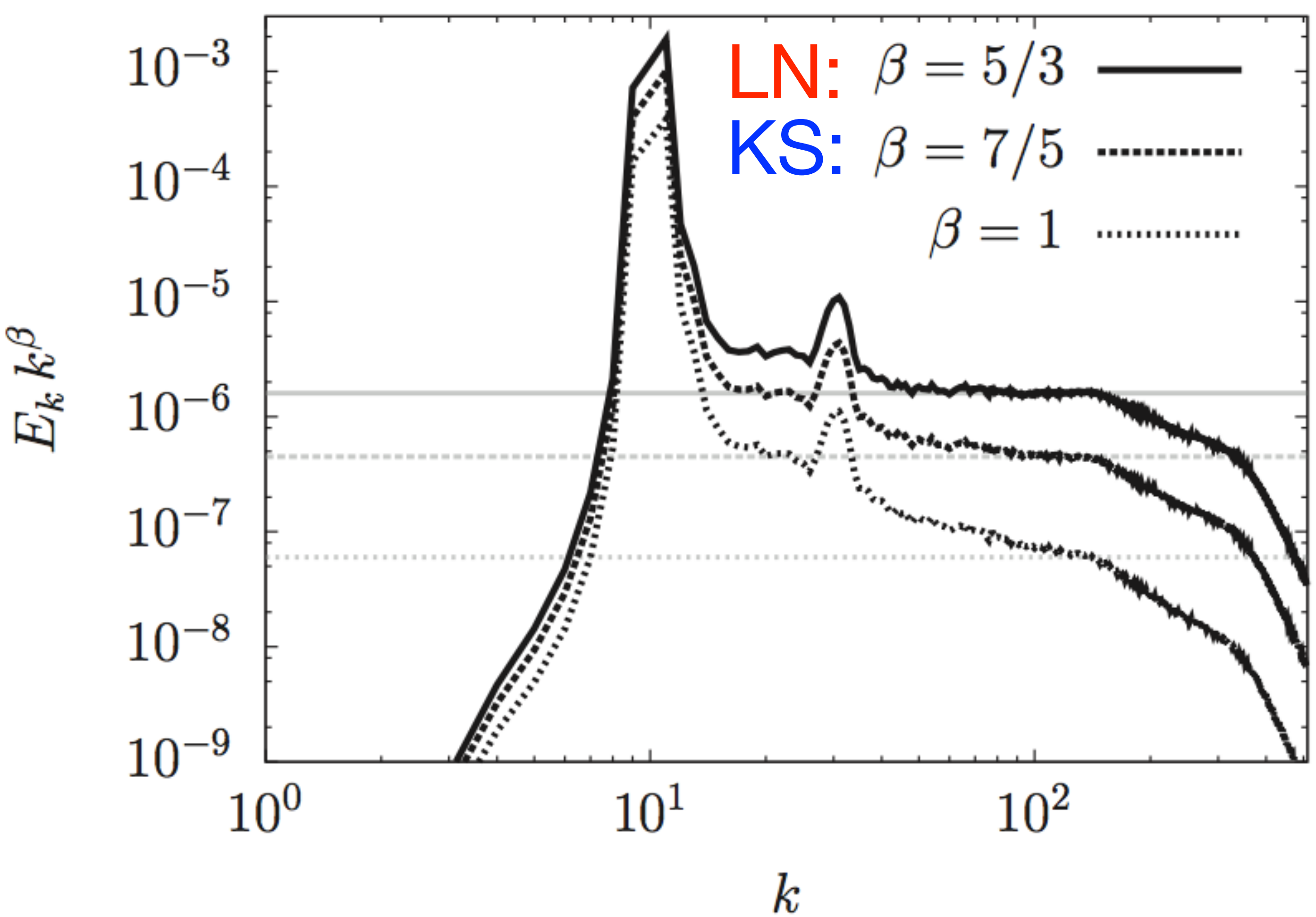
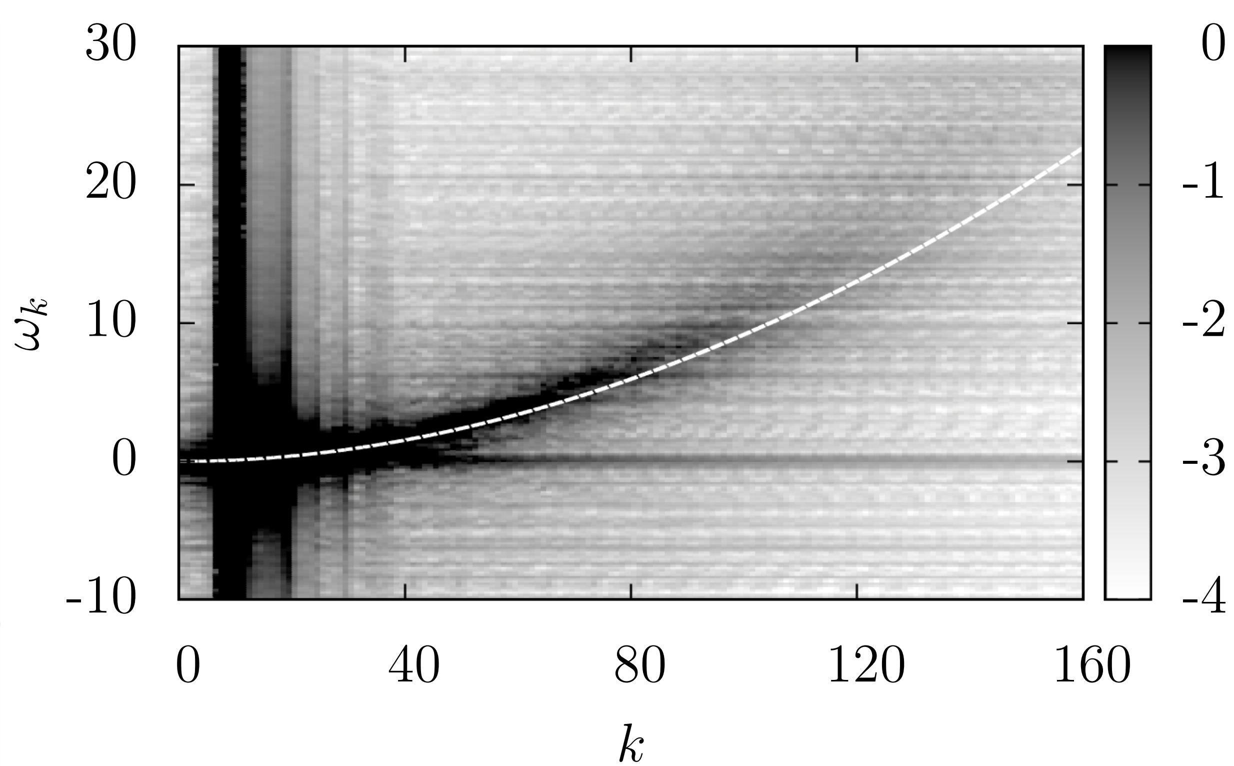


Spatio-temporal spectrum



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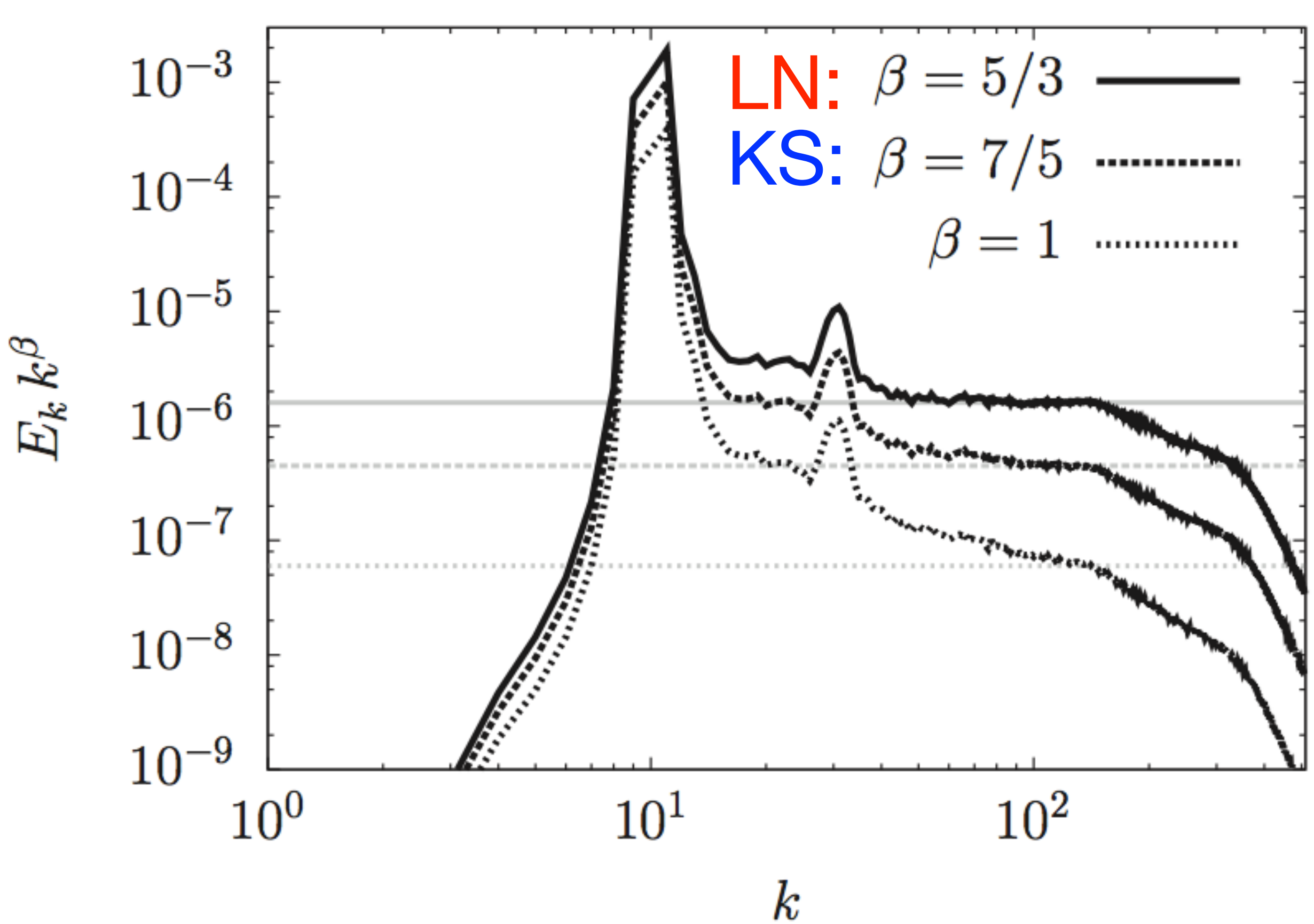
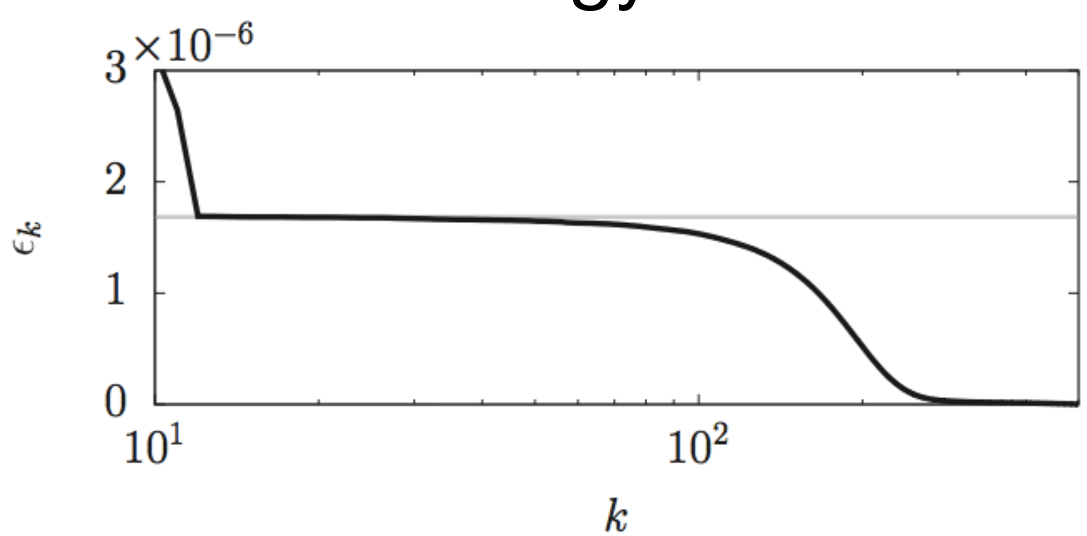


Energy flux



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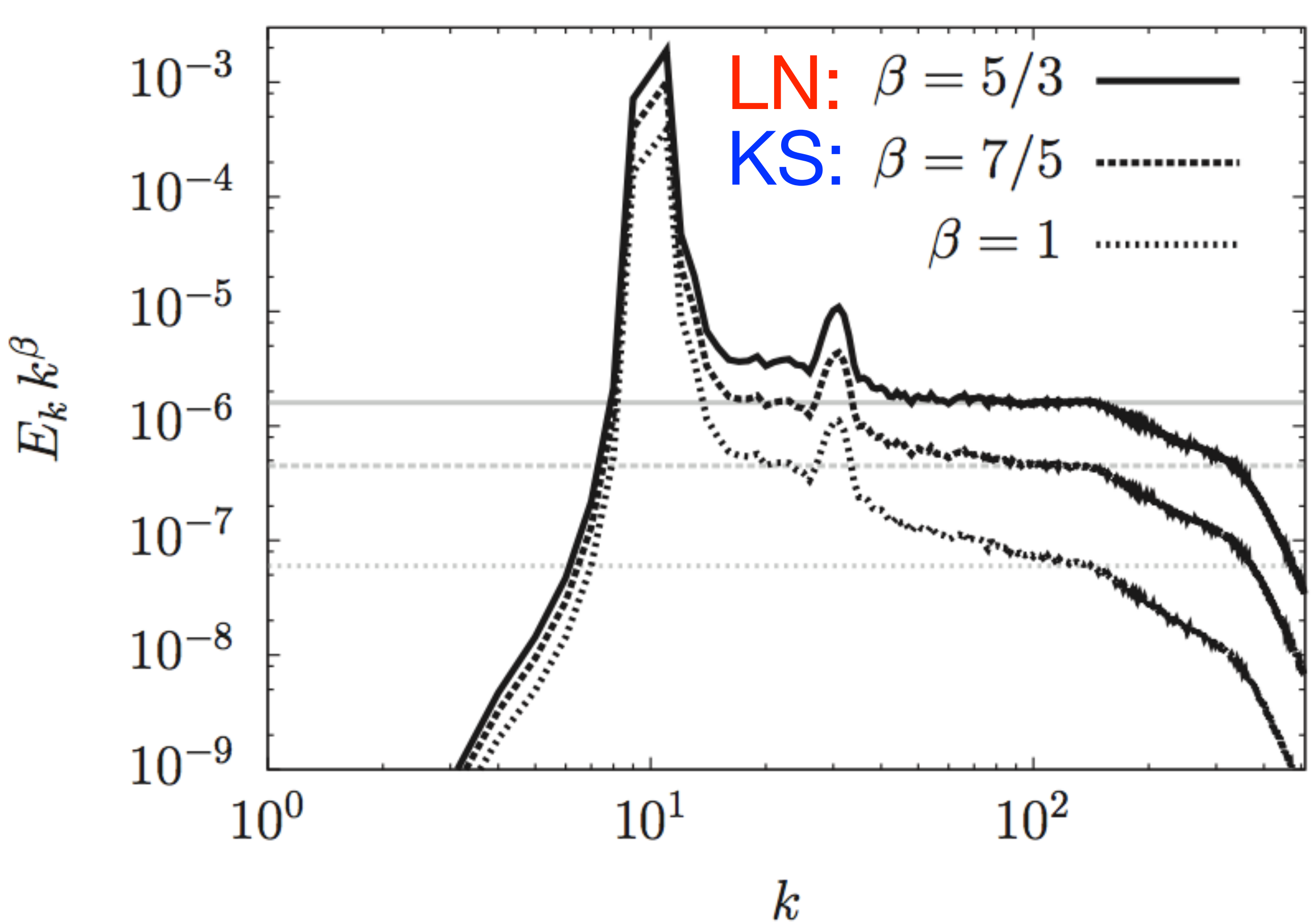

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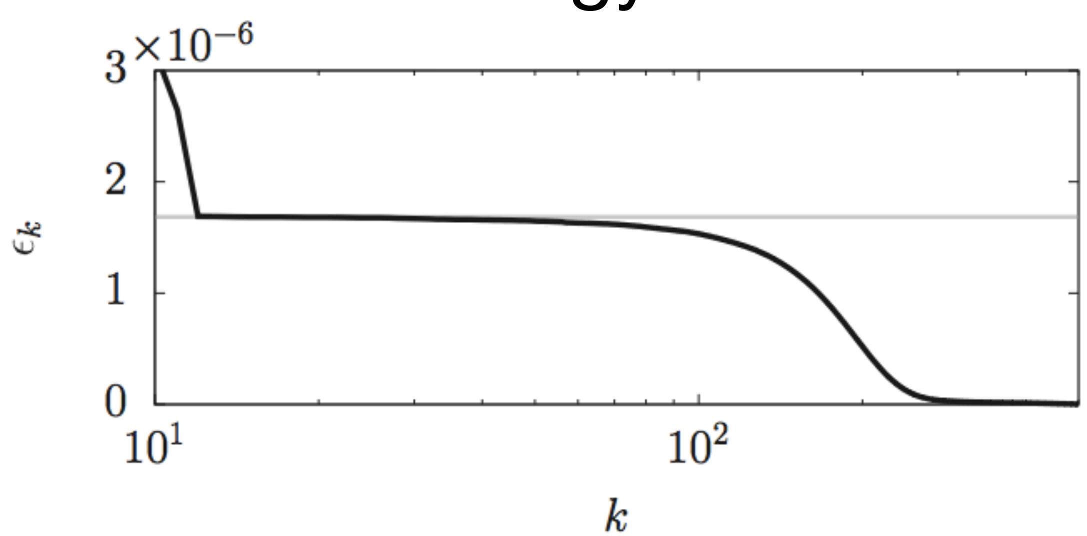

Measured prefactors

$C_{LN}^{num} = 0.318 \qquad C_{KS}^{num} = 0.0087$

- Within 5% of theoretical $C_{LN} = 0.304$

Identification of Spectrum

Aston University

Gross-Pitaevskii equation

$$i\dot{\Psi} = -\nabla^2\Psi + \Psi\left|\Psi\right|^2$$

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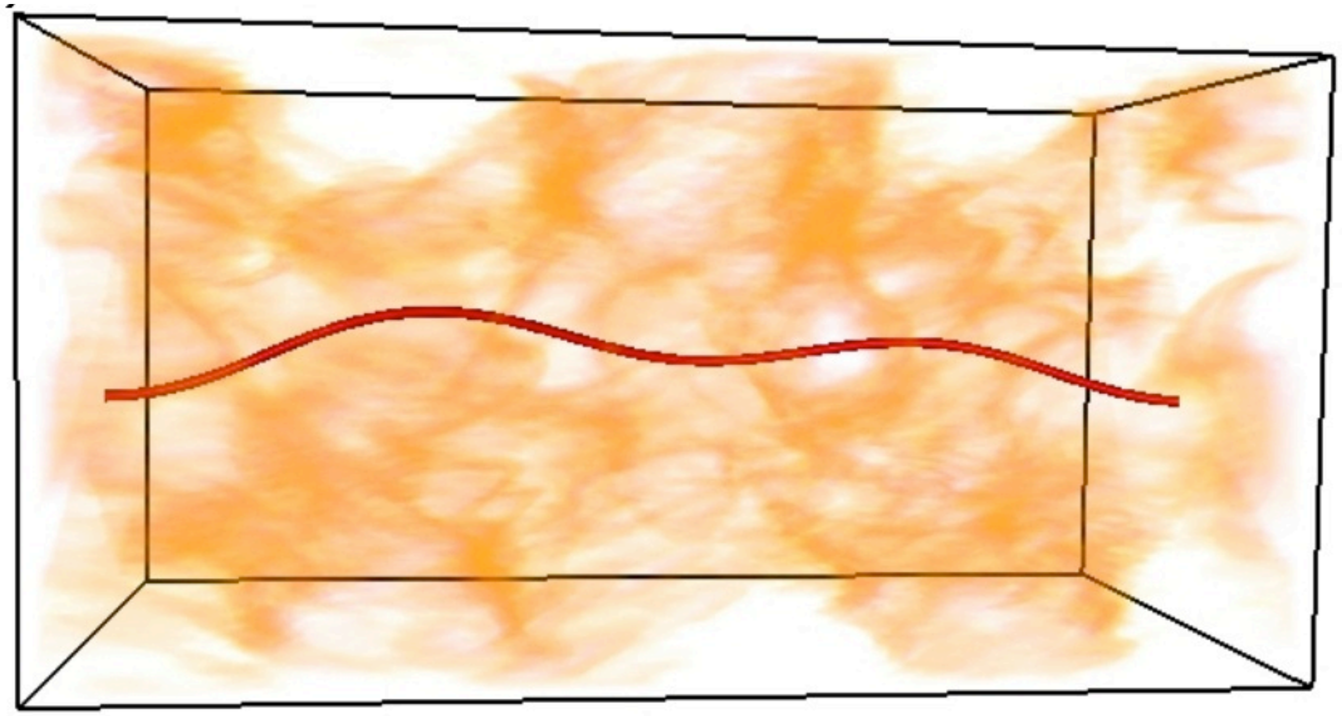

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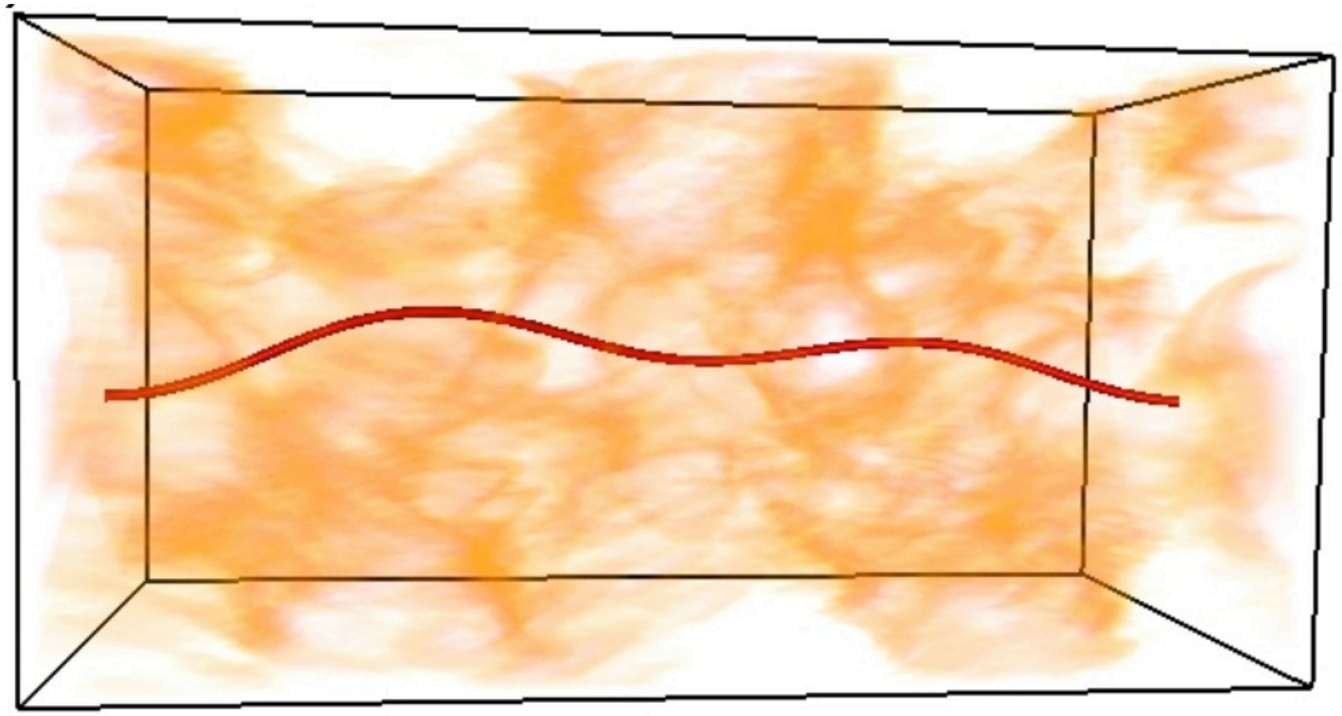

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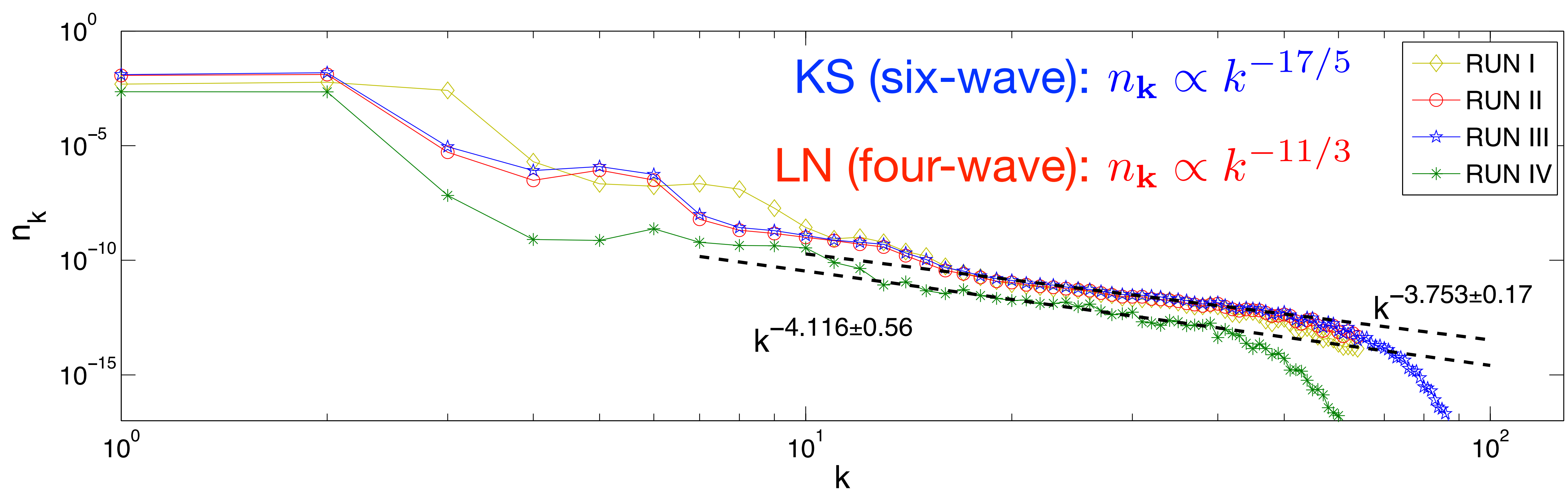

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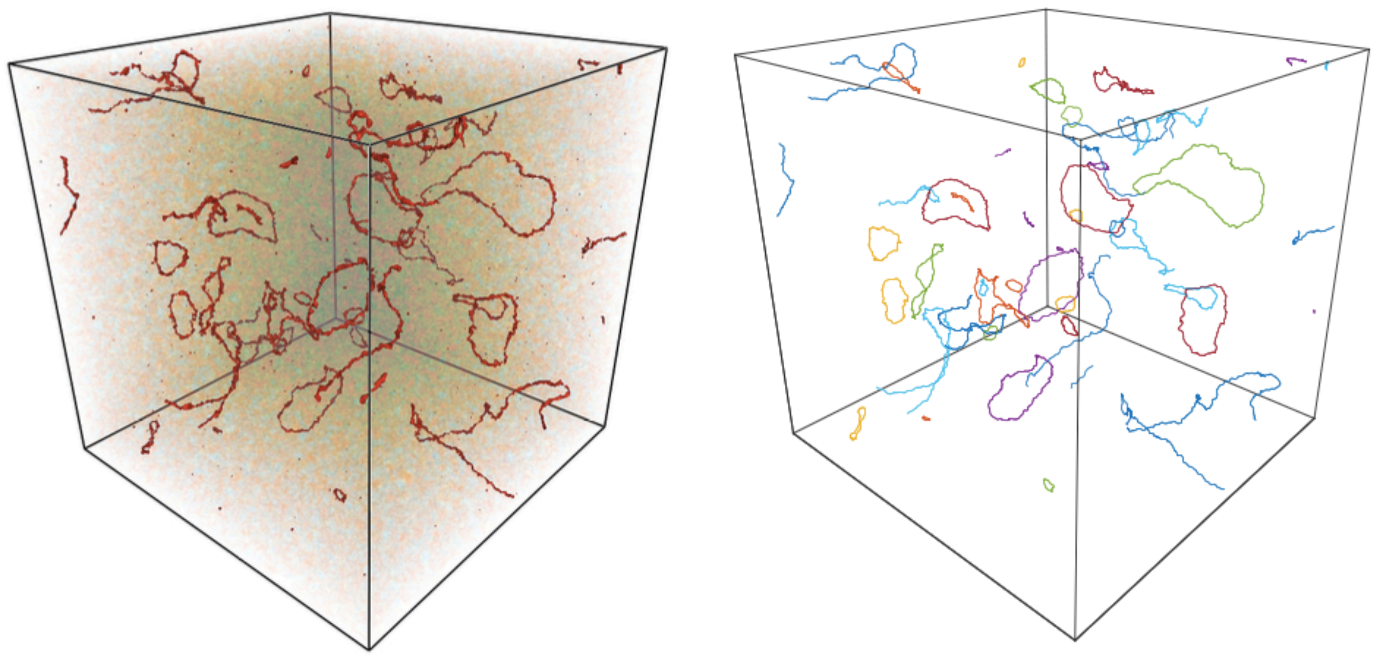

Villois, Phys. Rev. E, **93**, 061103(R), (2016)

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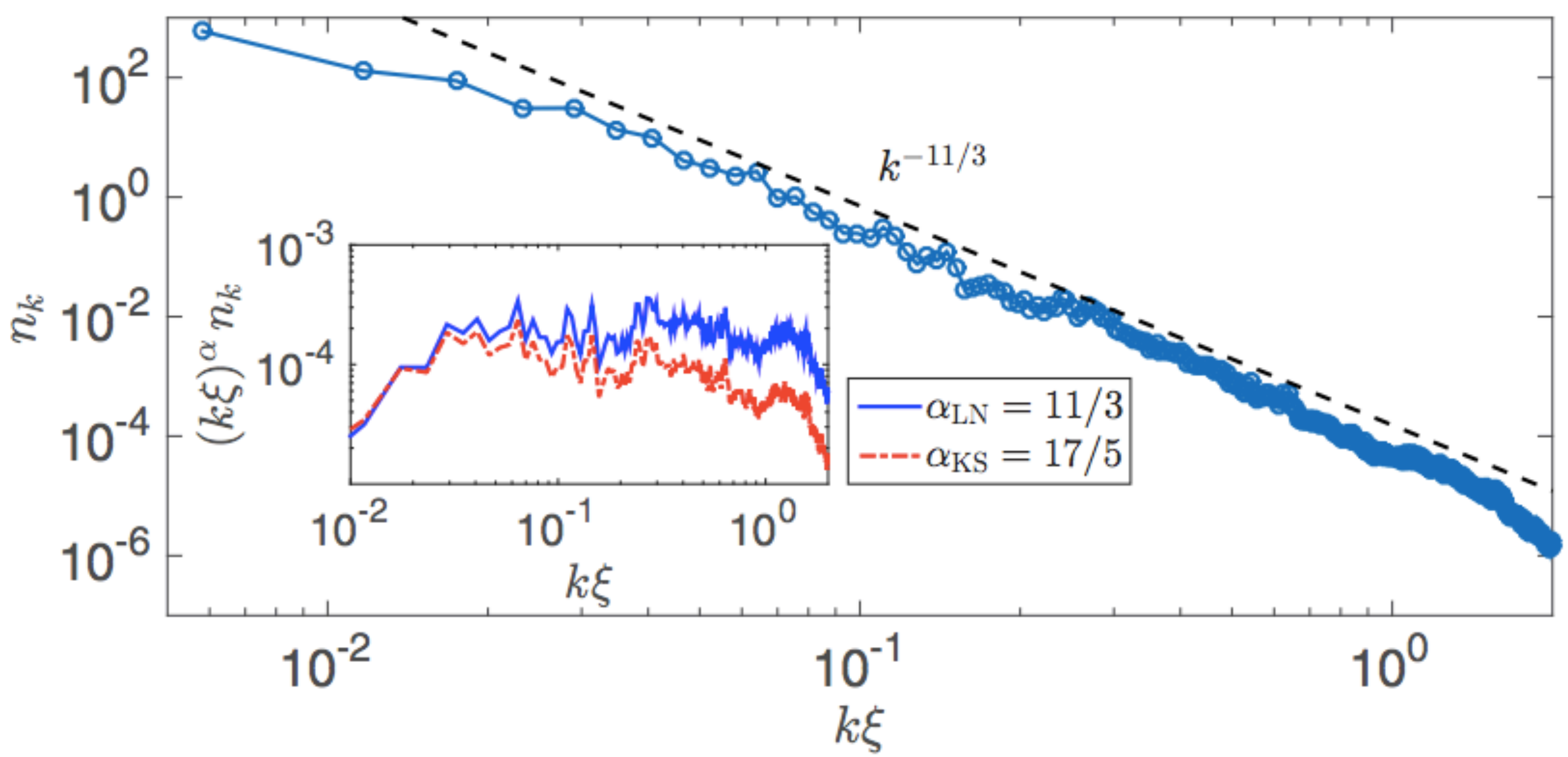

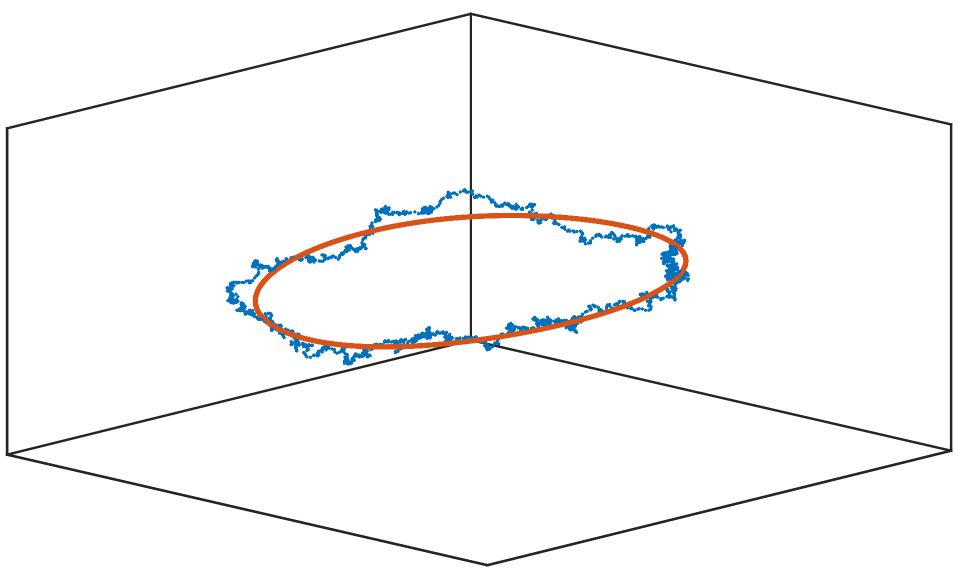

Conclusions and Perspectives

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Perspectives

- Can we quantify the amount of energy transferred to Kelvin waves?
- Are Kelvin-waves weakly nonlinear in reality?
- Observation of Kelvin-wave cascade in *velocity energy spectrum*?
- UK Fluids Network SIG in Wave Turbulence focussed on *strong* nonlinearities in WT