Waves And Complexity
Existence and Stability of Solitary Waves

Gideon Simpson

Department of Mathematics
Drexel University

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Introduction to NLS/GP

Nonlinear Schrödinger/Gross-Pitaevksii

$$i\partial_t\phi = -\nabla^2\phi + V(x)\phi + f(|\phi|^2)\phi = 0, \quad \phi : \mathbb{R}^{d+1} \to \mathbb{C} \tag{1.1}$$

See monographs:

- Sulem & Sulem (99), [20]
- Cazenave (03), [4]
- Fibich (14), [8]

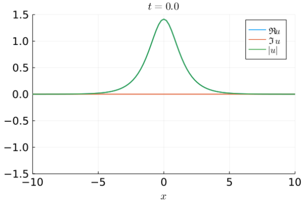
Often studied over $\mathbb{T}^d$, particularly in numerical simulations

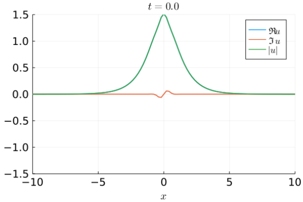
"Classical" Focusing Case

$$i\partial_t\phi = -\nabla^2\phi - |\phi|^{2\sigma}\phi = 0, \quad \phi : \mathbb{R}^{d+1} \to \mathbb{C} \tag{1.2}$$

Solitary Waves and their Stability

Cubic NLS in 1D





Structure of NLS/GP

Hamiltonian Flow

$$\partial_t \phi = -i D_{\bar{\phi}} \mathcal{H} \tag{1.3}$$

$$\mathcal{H}[\phi] = \int |\nabla \phi|^2 + V(x)|\phi|^2 + F(|\phi|^2) \tag{1.4}$$

and $F' = f$

Other Invariants

Mass/Power/Particle #/L^2:

$$\mathcal{N}[\phi] = \int |\phi|^2 \tag{1.5}$$

Also, momentum

Symmetries

$$\phi(x, t) \mapsto e^{i\gamma_0} \phi(x + x_0, t + t_0) \tag{1.6}$$

Additional symmetries when $V = 0$ and $f(s) = \pm s^\sigma$ (Dilation and Galilean)

Function Spaces, [14, 7]

Lebesgue spaces For $1 \leqslant p < \infty$

$$L^p(\mathbb{R}^d) = \left\{ f \mid \left\{ \int |f(x)|^p \right\}^{1/p} < \infty \right\} \tag{1.7}$$

and for $p = \infty$

$$L^\infty(\mathbb{R}^d) = \{ f \mid \mathrm{esssup}_x |f(x)| < \infty \} \tag{1.8}$$

Sobolev space

$$H^1(\mathbb{R}^d) = \left\{ f \mid \sqrt{\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2} < \infty \right\} \tag{1.9}$$

Sobolev Inequalities

Gagliardo-Nirenberg Inequality, $d \geqslant 2$

For

$$\sigma < \frac{2}{d-2}, \tag{1.10}$$

we have

$$\|f\|_{L^{2\sigma+2}}^{2\sigma+2} \lesssim \|\nabla f\|_{L^2}^{\sigma d} \|f\|_{L^2}^{2+\sigma(2-d)} \lesssim \|f\|_{H^1}^{2\sigma+2} \tag{1.11}$$

Dimension $d = 1$

$\|f\|_{L^\infty} \lesssim \|f\|_{H^1}$, so

$$\|f\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant \|f\|_{L^\infty}^{2\sigma} \|f\|_{L^2}^{2} \lesssim \|f\|_{H^1}^{2\sigma} \|f\|_{L^2}^{2} \lesssim \|f\|_{H^1}^{2\sigma+2} \tag{1.12}$$

Solitary Waves

- Solitary Wave Ansatz:

$$\phi(x,t) = e^{i\omega t} R(x;\omega), \tag{1.13}$$

- Solitary wave PDE:

$$\omega R - \nabla^2 R + V(x)R + f(|R|^2)R = 0 \tag{1.14}$$

  with $\omega > 0$, and R is the unknown
- Alternatively, fixing the 2-norm (Mass/Power), (R,ω) is the solution of a nonlinear eigenvalue problem

Simplification

- "Classical" focusing case with $V = 0$ and $f(s) = -s^\sigma$:

$$\omega R - \nabla^2 R - |R|^{2\sigma} R = 0 \tag{2.1}$$

- "Subcritical" regime:

$$0 < \sigma < \frac{2}{d-2} \tag{2.2}$$

Dimension One – First Integrals

Assume R is real valued

$$\omega R - R'' - R^{2\sigma+1} = 0 \tag{2.3}$$

- Multiply by R' and integrate:

$$\frac{\omega}{2}R^2 - \frac{1}{2}(R')^2 - \frac{1}{2\sigma+2}R^{2\sigma+2} = K \tag{2.4}$$

- Under the assumption that $R, R' \to 0$ at $\pm\infty$, $K = 0$

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- Rewrite as

$$-\frac{dR}{\sqrt{\omega R^2 - \frac{1}{\sigma+1}R^{2\sigma+2}}} = dx \tag{2.5}$$

- Infer that the peak ($x = 0$), where $R' = 0$, is

$$R(0) = [\omega(\sigma+1)]^{\frac{1}{\sigma+1}} \tag{2.6}$$

Solution

- From table of integrals/Mathematica/MAPLE/etc.:

$$R = [\omega(\sigma + 1)]^{\frac{1}{2\sigma}} \operatorname{sech}^{\frac{1}{\sigma}}(\sigma\sqrt{\omega}x) \tag{2.7}$$

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$$\mathcal{H}[\phi_g] \approx A^2\alpha^d \left\{\alpha^{-2} - A^{2\sigma}\right\} \tag{2.8}$$

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- **But**, we also have $\mathcal{N}[\phi_g] \propto A^2\alpha^d$

Solitary Waves as Constrained Minimizers

- Consider minimizing $\mathcal{H}$ subject to the constraint $\mathcal{N} = N$
- Lagrange multiplier problem:

$$\min_{(\phi,\lambda)} \mathcal{H}[\phi] + \lambda(\mathcal{N}[\phi] - N) \tag{2.9}$$

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- Euler-Lagrange equation:

$$-\nabla^2\phi - |\phi|^{2\sigma}\phi + \lambda\phi = 0 \tag{2.10}$$

- Under the identifications $\phi = R$ and $\lambda = \omega$, we have the solitary wave equation (again)

Rescaling – Eliminating the Parameter

- Let
$$R(x) = \omega^{\frac{1}{2\sigma}} \tilde{R}(\sqrt{\omega} x) \tag{2.11}$$
- $\tilde{R}$ solves
$$-\nabla^2 \tilde{R} - |\tilde{R}|^{2\sigma} \tilde{R} + \tilde{R} = 0 \tag{2.12}$$
- Focus on $\omega = 1$ case
- External potentials and other nonlinearities break scaling

Optimal Gagliardo-Nirenberg Constant

An Alternative Variational Problem, Weinstein (83), [21, 20, 8])

$$J[f] = \frac{\|\nabla f\|_{L^2}^{\sigma d}\|f\|_{L^2}^{2+\sigma(2-d)}}{\|f\|_{L^{2\sigma+2}}^{2\sigma+2}} \tag{2.13}$$

J is defined over $f \in H^1(\mathbb{R}^d)$, $f \neq 0$.
Consider the variational problem:

$$\inf_{f \in H^1, f \neq 0} J[f] \tag{2.14}$$

The infinum, $C_{\sigma,d} > 0$, is the **optimal** constant in the Gagliardo-Nirenberg inequality:

$$\|f\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_{\sigma,d}\|\nabla f\|_{L^2}^{\sigma d}\|f\|_{L^2}^{2+\sigma(2-d)} \tag{2.15}$$

Optimal Constant and Solitary Waves

Theorem

The infinum of J is obtained at $f_\star$, a real valued, non-negative, and radially symmetric function.
$f_\star$ may be rescaled to correspond to $R = R_1$, the solitary wave with $\omega = 1$.
The optimal constant:

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$$R(r) \asymp r^{-\frac{d-1}{2}} e^{-r} \tag{2.16}$$

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Asymptotic Decay and Simulation

Important for constructing artificial radiation boundary conditions in numerical simulation in a finite domain

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or

$$\left(1 - \left(\frac{\mathcal{N}[\psi]}{\mathcal{N}[R]}\right)^{2/d}\right)\|\nabla\psi\|_{L^2}^2 \leqslant \mathcal{H}[\psi] \tag{2.18}$$

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- Anticipate singularities for large data

Prior Bounds

- Obviously, $J[f] \geqslant 0$
- By earlier work by Gagliardo-Nirenberg, there exists (non optimal) $C > 0$ such that
$$J[f] \geqslant \frac{1}{C} > 0$$
for all $f \in H^1$, $f \neq 0$

Minimizing Sequences

- Let $f_n \in H^1$, $f_n \neq 0$, be a **minimizing sequence** of $J[f_n]$:

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- We establish:
  - If $f_\star$ (a minimizer) exists, it can be taken to be real valued, so we may assume the f_n are real
  - We may take the $f_n \geqslant 0$ and radial
  - The f_n have a subsequential limit in H^1: $f_\star$

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- If a minimizer exists, $f_\star(x) = A(x)e^{i\theta}$; take $\theta = 0$

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- If a minimizer exists, $f_\star(x) = A(x)e^{i\theta}$; take $\theta = 0$
- We may take the minimizing sequence to be real valued

Non-negativity of the Minimizing Sequence

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- We may replace f_n with $|f_n|$ and relabel: $f_n \geqslant 0$

Symmetrization of the Minimizing Sequence

- Steiner symmetrization: for each f_n, there exists $\tilde{f}_n(x) = \tilde{f}_n(|x|)$, a radial function such that:

$$\|\tilde{f}_n\|_{L^p} = \|f_n\|_{L^p}$$
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- Consequently, $J[f_n] \geqslant J[\tilde{f}_n]$
- Replace f_n with $\tilde{f}_n$ and relabel: a sequence of non-negative, real valued, radial functions

Rescaling

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- **Claim:** J is invariant to this scaling:

$$J[f^{\lambda,\mu}] = J[f] \tag{2.21}$$

- Different norms scale differently:

$$\|f^{\lambda,\mu}\|_{L^2}^2 = \mu^2 \lambda^{-d} \|f\|_{L^2}^2 \tag{2.22}$$

$$\|\nabla f^{\lambda,\mu}\|_{L^2}^2 = \mu^2 \lambda^{2-d} \|\nabla f\|_{L^2}^2 \tag{2.23}$$

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- For each f_n, there exist (λ_n, μ_n) such that:

$$\|f_n^{\lambda_n,\mu_n}\|_{L^2}^2 = \|\nabla f_n^{\lambda_n,\mu_n}\|_{L^2}^2 = 1 \tag{2.24}$$

- Replace f_n with $f_n^{\lambda_n,\mu_n}$ and relabel

Extracting the Limit

- We have $f_n = f_n(|x|) \geqslant 0$ with $\|f_n\|_{L^2}^2 = \|\nabla f_n\|_{L^2}^2 = 1$ and

$$\lim_{n\to\infty} J[f_n] = \lim_{n\to\infty} \frac{1}{\|f_n\|_{L^{2\sigma+2}}^{2\sigma+2}} = \inf J[f]$$

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- By Fatou's lemma, and reindexing as f_n,

$$\|f_\star\|_{L^2} \leqslant \liminf \|f_n\|_{L^2} = 1$$
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- For radial functions, H^1 embeds **compactly** into $L^{2\sigma+2}$ $(\sigma < 2/(d-2))$: there exists a subsequence of f_n that **strongly** converges in $L^{2\sigma+2}$ to $f_\star$

Extracting the Limit, Continued

- We now have

$$\inf J[f] \leqslant J[f_\star] \leqslant \frac{1}{\|f_\star\|_{L^{2\sigma+2}}^{2\sigma+2}} = \lim_{n\to\infty} \frac{1}{\|f_n\|_{L^{2\sigma+2}}^{2\sigma+2}} = \lim_{n\to\infty} J[f_n] = \inf J[f]$$

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- The infinimum is obtained at $f_\star$, non-negative and radial
- Additionally, $\|f_\star\|_{L^2} = \|\nabla f_\star\|_{L^2} = 1$

Euler-Lagrange Equations

- At a critical point f (i.e., a minimizer):

$$DJ[f] = -\frac{\sigma d}{\|\nabla f\|_{L^2}^2}\nabla^2 f + \frac{2+\sigma(2-d)}{\|f\|_{L^2}^2} f - \frac{2\sigma+2}{\|f\|_{L^{2\sigma+2}}^{2\sigma+2}}|f|^{2\sigma} f = 0$$

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- At $f_\star$,

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- Under another rescaling, this is

$$-\nabla^2 R + R - R^{2\sigma+1} = 0$$

Additional Remarks

- Same approach can be used to obtain existence of ground states with potential – Rose & Weinstein (88), [19, 20]
- Supercritical case, $\sigma > 2/(d-2)$, does **not** have solitary waves – application of the Pohozaev identities
- Dark solitons – in settings where $|\phi| \to 1$ at ∞, there are solitary waves

Uniqueness of the Ground State

- Under the conclusion of radial symmetry, solitary wave equation becomes an ODE:

$$-R'' - \frac{d-1}{r}R' + R - R^{2\sigma+1} = 0 \tag{2.25}$$

- By uniqueness of solutions of ODEs, we can conclude uniqueness of the ground state – see Kwong (89), [13], McLeod & Serrin (87), [16], and Coffman (72), [6]

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- (2.25) has other solutions for $d \geqslant 2$ – **excited states**, with a nonzero number of zero crossings

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$$-R'' - \frac{d-1}{r}R' + R - R^{2\sigma+1} = 0 \qquad (2.25)$$

- By uniqueness of solutions of ODEs, we can conclude uniqueness of the ground state – see Kwong (89), [13], McLeod & Serrin (87), [16], and Coffman (72), [6]
- (2.25) has other solutions for $d \geqslant 2$ – **excited states**, with a nonzero number of zero crossings
- Another class of exicited states take the form $R(x) = \rho(r)S(\boldsymbol{\theta})$

Scalar Problem

Hamiltonian Flow

For

$$H(q, p) = \frac{1}{2}p^2 + V(q) \tag{3.1}$$

consider the Hamiltonian flow:

$$\dot{q} = H_p = p \tag{3.2a}$$

$$\dot{p} = -H_q = -V'(q) \tag{3.2b}$$

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Stationary Solutions

Assume (smooth) V has (local) minimum $q_\star$, making $x_\star \equiv (q_\star, 0)$ a stationary solution of (3.2) – is it stable?

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Motivation

Much of the intuition and methodology for this problem carries over to NLS and related problems

Scalar Stability

- **Goal**: Use the invariance of H to get stability of the stationary solution
- $x_\star$ will said to be a stable solution of the dynamical system $x' = JDH(x)$ provided: for all $\epsilon > 0$, there exists $\delta > 0$, such that

$$|x_0 - x_\star| \leqslant \delta \Rightarrow |x(t) - x_\star| \leqslant \epsilon \tag{3.3}$$

for all t

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  for all t
- In finite dimensions, all norms are equivalent – use whichever is convenient
- This is **not** asymptotic stability; $x(t)$ need not converge to $x_\star$

Scalar Stability, Continued

Taylor expanding:

$$
\begin{aligned}
\Delta H = H(q,p) - H(q_\star, 0) &= H(q_\star + \delta q, \delta p) - H(q_\star, 0) \\
&= \frac{1}{2}\delta p^2 + V'(q_\star)\delta q + \frac{1}{2}V''(q_\star)\delta q^2 + \dots \qquad (3.4)\\
&= \frac{1}{2}\delta p^2 + \frac{1}{2}V''(q_\star)\delta q^2 + \dots
\end{aligned}
$$

Scalar Stability, Continued

Taylor expanding:

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- Since $q_\star$ is a (local) minimizer of V, $V''(q_\star) > 0$

Scalar Stability, Continued

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- Since $q_\star$ is a (local) minimizer of V, $V''(q_\star) > 0$
- To leading order, we have a prior bound on $(\delta q, \delta p)$
- Not rigorous (yet)

Scalar Stability, Rigorous Analysis

Theorem

If V is smooth and $V''(q_\star) > 0$, then $(q_\star, 0)$ is stable

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- Employ Taylor's theorem with remainder:

$$
\begin{aligned}
V(q_\star + \delta q) &= V(q_\star) + \frac{1}{2} V''(q_\star)\delta q^2 \\
&\quad + \frac{1}{2}\delta q^3 \int_0^1 (1-\tau)^2 V'''(q_\star + \tau \delta q) d\tau
\end{aligned}
\tag{3.5}
$$

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If V is smooth and $V''(q_\star) > 0$, then $(q_\star, 0)$ is stable

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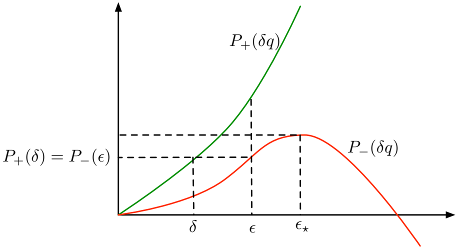
- Assuming $|\delta q| \leqslant 1$, there exist $C, D > 0$, such that

$$\frac{1}{2}\delta p^2 + \underbrace{\frac{1}{2}V''(q_\star)\delta q^2 - C\delta q^3}_{\equiv P_-(\delta q)} \leqslant \Delta H \leqslant \frac{1}{2}\delta p^2 + \underbrace{\frac{1}{2}V''(q_\star)\delta q^2 + D\delta q^3}_{\equiv P_+(\delta q)} \tag{3.6}$$

Scalar Stability, Rigorous Analysis, Continued



- $\epsilon_\star$ (assumed $\leqslant 1$)
- $\epsilon \leqslant \epsilon_\star$; δ is the value such that $P_+(\delta) = P_-(\epsilon)$
- **Geometric Idea:** Remain in region where $P_\pm$ are both monotonic increasing

Scalar Stability, Rigorous Analysis, Continued

- Assume data satisfies:

$$|\delta q_0| \leqslant \delta \tag{3.7a}$$

$$\tfrac{1}{2}\delta p_0^2 + P_+(\delta q_0) \leqslant P_-(\delta) \tag{3.7b}$$

- **Claim:**

$$|\delta q(t)| \leqslant \epsilon \tag{3.8a}$$

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Scalar Stability, Rigorous Analysis, Continued

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- **Proof by Contradiction:**
- Suppose $\epsilon < |\delta q(t)| \leqslant \epsilon_c$, then using the polynomial bounds:

$$\begin{aligned}\Delta H(t) &\geqslant \tfrac{1}{2}\delta p(t)^2 + P_-(\delta q(t)) \geqslant P_-(\delta q(t)) \\ &> P_-(\epsilon) > P_-(\delta) \geqslant \tfrac{1}{2}\delta p_0^2 + P_+(\delta q_0) \geqslant \Delta H_0\end{aligned} \tag{3.9}$$

Scalar Stability, Rigorous Analysis, Continued

- Now suppose $|\delta q(t)| \leqslant \epsilon$, but

$$P_{-}(\epsilon) < \tfrac{1}{2}\delta p(t)^2 + P_{-}(\delta q(t)) \tag{3.10}$$

- Then

$$\begin{aligned} \Delta H_0 &\leqslant \frac{1}{2}\delta p_0^2 + P_{+}(\delta q_0) \leqslant P_{-}(\delta) < P_{-}(\epsilon) \\ &< \frac{1}{2}\delta p(t)^2 + P_{-}(\delta q(t)) \leqslant \Delta H(t) \end{aligned} \tag{3.11}$$

- Consequently, $(\delta q(t), \delta p(t))$ will stay within the ϵ neighborhood of $(0, 0)$
- This relies on the solution, $(q(t), p(t))$ being a continuous:

$$(q(t), p(t)) \in C(0, \infty; \mathbb{R}^2). \tag{3.12}$$

We omit the details

Generalized Korteweg–de Vries (gKdV)

$$u_t + u^p u_x + u_{xxx} = 0, \quad p \geqslant 1 \tag{4.1}$$

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$$\mathcal{H}[u] = \int \frac{1}{2} u_x^2 - \frac{1}{(p+1)(p+2)} u^{p+2} dx \tag{4.2}$$

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$$\phi_c(\xi) = \left[c \frac{(p+1)(p+2)}{2}\right]^{\frac{1}{p}} \operatorname{sech}^{\frac{2}{p}} \left(\frac{\sqrt{c}p}{2} \xi\right), \quad \xi = x - ct - x_0 \tag{4.5}$$

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- Stability proof is simpler, but has same steps as NLS

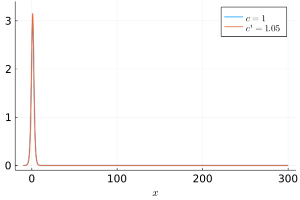
Background on Orbital Stability in (g)KdV

- Major result from Benjamin (1972), [1], inspired by Boussinesq (1877), [3]
- Result improved/corrected by Bona (1975), [2]
- Weinstein (1986) applied ideas developed NLS to gKdV, [23] – **approach presented here**
- Methodology of Grillakis, Shatah, and Strauss (1987, 1990) [10, 11]; see, also, Kapitula & Promislow (2013), [12]

Necessity of a New Metric

- Given a fixed $c > 0$ and $x_0 = 0$, consider the stability of ϕ_c
- Consider a slightly perturbed solitary wave, $\phi_{c'}$ with $c' > c$; for any of the "usual" norms (i.e., L^p or H^1),

$$\lim_{c' \to c} \|\phi_c - \phi_{c'}\| \to 0 \tag{4.6}$$



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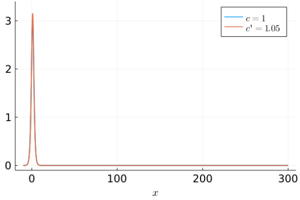

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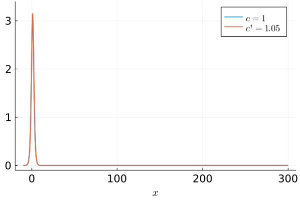

- Need a new metric

Sliding Metric and Orbital Stability

- Introduce the "sliding" metric

$$\mathrm{d}(f, g) = \inf_y \|f - g(\cdot + y)\|_{H^1} = \inf_y \|g - f(\cdot + y)\|_{H^1} \tag{4.7}$$

- Removes the spatial translation symmetry of the problem

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- This is equivalent to

$$\mathrm{d}(f,g) = \inf_{\tilde{g}\in\mathcal{O}(g)} \|f - \tilde{g}\|_{H^1} \tag{4.8}$$

where $\mathcal{O}(g)$ is the **orbit** of g under the translation symmetry group:

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- ϕ_c will be **orbitally stable** if

$$\mathrm{d}(u_0, \phi_c) \leqslant \delta \Rightarrow \mathrm{d}(u(t), \phi_c) \leqslant \epsilon \tag{4.10}$$

Sliding Metric, Continued

- Slight generalization of the metric: For $c > 0$

$$\mathrm{d}_c(f, g) = \inf_y \sqrt{\|f' - g'(\cdot + y)\|_{L^2}^2 + c\|f - g(\cdot + y)\|_{L^2}^2} \qquad (4.11)$$

- From previous slide, $\mathrm{d} = \mathrm{d}_1$

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- From previous slide, $\mathrm{d} = \mathrm{d}_1$
- These are all equivalent distances:

$$\sqrt{\min\{1, c\}}\, \mathrm{d}_1(f, g) \leqslant \mathrm{d}_c(f, g) \leqslant \sqrt{\max\{1, c\}}\, \mathrm{d}_1(f, g) \qquad (4.12)$$

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- For stability of ϕ_c, we will prove d_c, the **physical** metric, remains small, and infer d_1, the **mathematical** metric, remains small

Orbital Stability of gKdV

Theorem

For $p < 4$, and all $c > 0$, the gKdV solitary wave is orbitally stable. For all $\epsilon > 0$, there exists $\delta > 0$, such that

$$\mathrm{d}_c(u_0, \phi_c) \leqslant \delta \Rightarrow \mathrm{d}_c(u(t), \phi_c) \leqslant \epsilon, \quad t \geqslant 0.$$

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Strategy of proof:

- Decompose $u(t)$ into the solitary wave an a perturbation
- Form a linear combination of the invariants and Taylor expand them about the solitary wave
- Show that a certain quadratic form is positive and bounded by these invariants such that the perturbation is bounded in terms of the data

Decomposition

- At time t, the optimal $x_0 = x_0(t)$ minimizes d_c:

$$\begin{aligned}\mathrm{d}_c(u(t), \phi_c)^2 = \|\partial_x \phi_c - \partial_x u(\cdot + x_0(t), t))\|_{L^2}^2 \\ + c\|\phi_c - u(\cdot + x_0(t), t))\|_{L^2}^2\end{aligned} \tag{4.13}$$

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- Decompose as:

$$u(x + x_0(t), t) = \phi_c(x) + v(x, t) \tag{4.14}$$

so

$$d_c(u(t), \phi_c)^2 = \|\partial_x v\|_{L^2}^2 + c\|v\|_{L^2}^2 \tag{4.15}$$

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- **Important:** v satisfies an orthogonality condition

$$\langle \phi_c^p \partial_x \phi_c, v\rangle = \int \phi_c^p \partial_x \phi_c v = 0 \tag{4.16}$$

Action Expansion

- Define the action

$$\mathcal{S}_c[u] \equiv \mathcal{H}[u] + c\mathcal{N}[u] \tag{4.17}$$

- Using $u(\cdot + x_0) = \phi_c + v$:

$$\mathcal{S}_c[u] = \mathcal{S}_c[\phi_c + v] \tag{4.18}$$

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- Taylor expand:

$$\begin{aligned}
\Delta\mathcal{S}_c(t) &= \mathcal{H}[v(t) + \phi_c] - \mathcal{H}[\phi_c] + c(\mathcal{N}[v(t) + \phi_c] - \mathcal{N}[\phi_c]) \\
&= \frac{1}{2}\int \left\{ v_x^2 + 2v_x\partial_x\phi_c + cv^2 + 2cv\phi_c \right\} dx \\
&\quad - \int \left\{ \frac{1}{p+1}\phi_c^{p+1} v + \frac{1}{2}\phi_c^p v^2 \right\} dx \\
&\quad - \int \left\{ v^3 \int_0^1 \frac{p}{2}(\phi_c + \tau v)^{p-1}(1-\tau)^2 d\tau \right\} dx
\end{aligned} \tag{4.19}$$

Action Expansion, Continued

- Grouping terms:

$$\Delta\mathcal{S}_c(t) = \frac{1}{2}\langle Lv, v\rangle - \underbrace{\int \left\{ v^3 \int_0^1 \frac{p}{2}(\phi_c + \tau v)^{p-1}(1-\tau)^2 d\tau \right\} dx}_{r_{c,p}[v]} \tag{4.20}$$

- Quadratic form $\langle Lv, v\rangle$:

$$L = -\partial_{xx} + c - \phi_c^p, \tag{4.21}$$

Schrödinger operator, self-adjoint on $L^2(\mathbb{R}^d)$

Bounding the Remainder

$$
\begin{aligned}
|r_{c,p}[v]| &= \left|\int \left\{ v^3 \int_0^1 \frac{p}{2}(\phi_c + \tau v)^{p-1}(1-\tau)d\tau \right\} dx \right| \\
&\lesssim \int |v|^3 \left\{ \int_0^1 (|\phi_c|^{p-1} + \tau |v|^{p-1}) d\tau \right\} dx \\
&\lesssim \|v\|_{L^3}^3 + \|v\|_{L^{p+2}}^{p+2}
\end{aligned}
\tag{4.22}
$$

Bounding the Remainder

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\begin{aligned}
|r_{c,p}[v]| &= \left| \int \left\{ v^3 \int_0^1 \frac{p}{2} (\phi_c + \tau v)^{p-1} (1-\tau) d\tau \right\} dx \right| \\
&\lesssim \int |v|^3 \left\{ \int_0^1 (|\phi_c|^{p-1} + \tau |v|^{p-1}) d\tau \right\} dx \\
&\lesssim \|v\|_{L^3}^3 + \|v\|_{L^{p+2}}^{p+2}
\end{aligned} \tag{4.22}
$$

- By Sobolev inequalities, for $q \geqslant 2$,

$$
\|v\|_{L^q} \lesssim \|v\|_{H^1}. \tag{4.23}
$$

- Hence there exist positive constants C and D such that

$$
|r_{c,p}[v]| \leqslant C\|v\|_{H^1}^3 + D\|v\|_{H^1}^{p+2} \tag{4.24}
$$

Upper Bound on the Quadratic Form

$$\begin{aligned} |\langle Lv, v\rangle| &= \left|\int (\partial_x v)^2 + (c - \phi_c^p) v^2\right| \\ &\lesssim \|\partial_x v\|_{L^2}^2 + \|v\|_{L^2}^2 \lesssim \|v\|_{H^1}^2 \end{aligned} \tag{4.25}$$

Thus, there exists $A > 0$, such that

$$|\langle Lv, v\rangle| \lesssim A\|v\|_{H^1}^2 \tag{4.26}$$

Reviewing Estimates so Far

- Using the upper bound on the quadratic form and the Taylor bound on the remainder:

$$
\begin{aligned}
&\frac{1}{2}\langle Lv, v\rangle - C\|v\|_{H^1}^3 - D\|v\|_{H^1}^{p+2}\\
&\leqslant \Delta\mathcal{S}_c(0) = \Delta\mathcal{S}_c(t) = \frac{1}{2}\langle Lv, v\rangle - r_{c,p}[v]\\
&\leqslant \frac{1}{2}A\|v\|_{H^1}^2 + C\|v\|_{H^1}^3 + D\|v\|_{H^1}^{p+2}
\end{aligned}
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\begin{aligned}
&\frac{1}{2}\langle Lv, v\rangle - C\|v\|_{H^1}^3 - D\|v\|_{H^1}^{p+2}\\
&\leqslant \Delta\mathcal{S}_c(0) = \Delta\mathcal{S}_c(t) = \frac{1}{2}\langle Lv, v\rangle - r_{c,p}[v]\\
&\leqslant \frac{1}{2}A\|v\|_{H^1}^2 + C\|v\|_{H^1}^3 + D\|v\|_{H^1}^{p+2}
\end{aligned}
$$

- **IF** $\langle Lv, v\rangle \geqslant B\|v\|_{H^1}^2 + \mathrm{O}(\|v\|_{H^1}^3)$ with $B > 0$, then

$$
\begin{aligned}
P_-(\|v\|)_{H^1} \leqslant \Delta\mathcal{S}_c(0) &\leqslant P_+(\|v\|_{H^1})\\
P_-(x) &= \frac{1}{2}Bx^2 - Cx^3 - Dx^{p+2}\\
P_+(x) &= \frac{1}{2}Ax^2 + Cx^3 + Dx^{p+2}
\end{aligned}
$$

Relationship to the Finite Dimensional Case

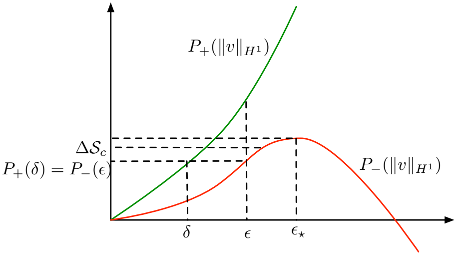
- Recall the finite dimensional bound, (3.6):

$$\frac{1}{2}\delta p^2 + P_-(\delta q) \leqslant \Delta H \leqslant \frac{1}{2}\delta p^2 + P_+(\delta q)$$

- We **almost** have:

$$P_-(\|v\|)_{H^1} \leqslant \Delta \mathcal{S}_c \leqslant P_+(\|v\|_{H^1})$$

Hypothetical Bound



Suppose, for all $\|v\|_{H^1}$ small enough:

$$P_-(\|v\|)_{H^1} \leqslant \Delta\mathcal{S} \leqslant P_+(\|v\|_{H^1})$$

and the data satisfies:

$$P_+(\|v_0\|)_{H^1} \leqslant P_-(\delta)$$

Assume At some time $\epsilon < \|v(t)\|_{H^1} \leqslant \epsilon_c$. Then:

$$\begin{aligned}\Delta\mathcal{S}_c(t) &\geqslant P_-(\|v(t)\|_{H^1}) > P_-(\epsilon) > P_-(\delta)\\ &\geqslant P_+(\|v_0)\|)_{H^1} \geqslant \Delta\mathcal{S}_c(0)\end{aligned}$$

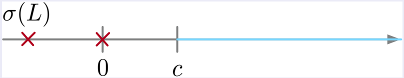
Contradiction $\|v(t)\|_{H^1} \leqslant \epsilon$ for all time

Linear Operator

Lemma

For the operator

$$L = -\partial_{xx} + c - \phi_c^p$$



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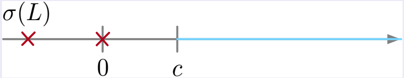

- Recall the solitary wave equation:

$$-\partial_{xxx}\phi_c + c\partial_x\phi_c - \phi_c^p\partial_x\phi_c = 0$$
$$L\partial_x\phi_c = 0$$

  so there is a zero eigenvalue
- $\partial_x\phi_c$ has one zero crossing – not the ground state
- Sturm-Liouville theory tells us there exists a negative eigenvalue corresponding to the ground state, $\psi_0 > 0$, of L – See Titschmarash ('46, '58) for proof on real line, also [12]

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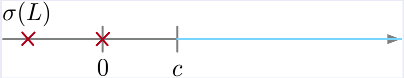

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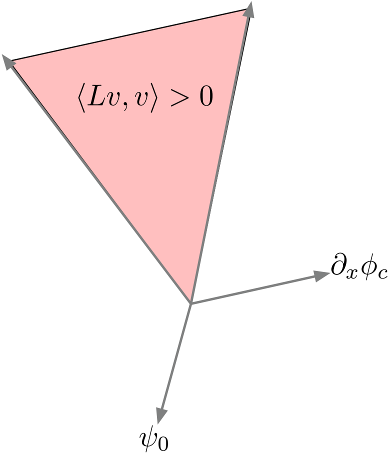
- Sturm-Liouville theory tells us there exists a negative eigenvalue corresponding to the ground state, $\psi_0 > 0$, of L – See Titschmarash ('46, '58) for proof on real line, also [12]
- For generic $v \in H^1$, $\langle Lv, v\rangle$ can be $\leqslant 0$

Geometry of the Quadratic Form



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- This introduces a **near orthogonality** condition:

$$\langle \phi_c, v \rangle = -\frac{1}{2} \|v\|_{L^2}^2 \tag{4.28}$$

- The $\mathcal{N}[\phi_c] = \mathcal{N}[u_0]$ condition can be relaxed

Quadratic Form Result

Proposition

If $\left\langle \phi_c^p \partial_x \phi_c, v \right\rangle = 0$, $\langle \phi_c, v\rangle = -\frac{1}{2}\|v\|_{L^2}^2$, *and* $\frac{d}{dc}\mathcal{N}[\phi_c] > 0$, *then*

$$\langle Lv, v\rangle \gtrsim \|v\|_{H^1}^2 - \|v\|_{H^1}^3 - \|v\|_{H^1}^4$$

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Strategy of Proof

1. Show $\langle Lv, v \rangle \geqslant 0$ under ideal orthogonality conditions

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4. Generalize to near orthogonality conditions

Non-negativity of the Quadratic Form

Proposition

If

$$\frac{d}{dc}\mathcal{N}[\phi_c] > 0$$

then

$$\alpha \equiv \inf_{\langle f,\phi_c\rangle=0,\|f\|_{L^2}=1} \langle Lf, f\rangle = 0$$

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$$\alpha \geqslant \inf_{\|f\|_{L^2}=1} \langle Lf, f\rangle = \lambda_0 > -\infty, \tag{4.29}$$

so $\alpha \in [\lambda_0, 0]$.

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Assume: $\alpha \in [\lambda_0, 0)$.

Non-negativity, Continued

Assuming $\alpha \in [\lambda_0, 0)$

Lagrange Multiplier Formulation

Using the method of Lagrange multipliers:

$$(L - \alpha)f_\star = \beta\phi_c$$

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Suppose $\beta = 0$. Then $\alpha < 0$ is an eigenvalue. But L has only one negative eigenvalue (by Sturm-Liouville) and $\alpha = \lambda_0$, so $f_\star = \psi_0 \geqslant 0$ is the ground state, but $\langle \psi_0, \phi_c \rangle \neq 0$, contradiction.

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Even with $\beta \neq 0$, $\alpha \neq \lambda_0$

If $\alpha = \lambda_0$,

$$0 = \langle f_\star, (L - \lambda_0 I)\psi_0 \rangle = \langle (L - \lambda_0 I) f_\star, \psi_0 \rangle = \beta \langle \phi_c, \psi_0 \rangle \neq 0$$

Non-negativity, Continued

$\beta \neq 0$, $\alpha \in (\lambda_0, 0)$

Spectral Function

For $\lambda \in (\lambda_0, 0]$,

$$g(\lambda) = \left\langle (L - \lambda I)^{-1} \phi_c, \phi_c \right\rangle, \tag{4.30}$$

$$g'(\lambda) = \|(L - \lambda I)^{-1} \phi_c\|_{L^2}^2 \geqslant 0, \tag{4.31}$$

non-decreasing

Non-negativity, Continued

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Properties of the Spectral Function

$$g(\alpha) = \left\langle (L - \alpha I)^{-1} \phi_c, \phi_c \right\rangle = \beta^{-1} \langle f_\star, \phi_c \rangle = 0.$$

and

$$g(0) = \left\langle L^{-1} \phi_c, \phi_c \right\rangle$$

Non-negativity, Continued

$\beta \neq 0$, $\alpha \in (\lambda_0, 0)$

- Recall the solitary wave equation:

$$-\partial_{xx}\phi_c + c\phi_c - \frac{1}{p+1}\phi_c^{p+1} = 0$$

Non-negativity, Continued

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- Recall the solitary wave equation:

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- Differentiate in c:

$$\begin{aligned} -\partial_{xx}\partial_c\phi_c + c\partial_c\phi_c - \phi_c^p\partial_c\phi_c &= -\phi_c \\ L\partial_c\phi_c &= \end{aligned} \tag{4.32}$$

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- Consequently

$$L^{-1}\phi_c = -\partial_c\phi_c + k\partial_x\phi_c$$

and

$$g(0) = \left\langle L^{-1}\phi_c, \phi_c \right\rangle = -\langle \partial_c\phi_c, \phi_c \rangle = -\tfrac{d}{dc}\mathcal{N}[\phi_c]$$

Non-negativity, Continued

$\beta \neq 0$, $\alpha \in (\lambda_0, 0)$

- $g(\lambda)$ is non-decreasing over $(\lambda_0, 0]$

Non-negativity, Continued

$\beta \neq 0$, $\alpha \in (\lambda_0, 0)$

- $g(\lambda)$ is non-decreasing over $(\lambda_0, 0]$
- If $\frac{d}{dc}\mathcal{N}[\phi_c] > 0$, then $g(0) < 0$; so $g(\lambda) < 0$ over $(\lambda_0, 0]$

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- But $g(\alpha) = 0$; contradiction

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- But $g(\alpha) = 0$; contradiction
- **Conclusion:** $\alpha \geqslant 0 \Rightarrow \alpha = 0$

Vakhitov-Kolokolov (VK) Condition

$$\frac{d}{dc}\mathcal{N}[\phi_c] > 0 \tag{4.33}$$

is a Vakhitov-Kolokolov condition; appears in NLS and other Hamiltonian equations with solitary waves

For $gKdV$,

$$\mathcal{N}[\phi_c] \propto c^{\frac{2}{p}-\frac{1}{2}},$$

so VK holds for $p < 4$

Getting Positivity

We have

$$\alpha \equiv \inf_{\langle f, \phi_c \rangle = 0, \|f\|_{L^2} = 1} \langle Lf, f \rangle = 0$$

but not positivity.

Proposition

If VK holds:

$$\eta \equiv \inf_{\langle f, \phi_c \rangle = 0, \langle f, \phi_c^p \partial_x \phi_c \rangle = 0, \|f\|_{L^2} = 1} \langle Lf, f \rangle > 0 \tag{4.34}$$

Proof: $\eta \geqslant \alpha = 0$. Suppose $\eta = 0$. Proceed with Lagrange Multipliers

Getting Positivity, Continued

$\eta = 0$

Lagrange Multipliers

$$Lf_\star = \lambda_1 \phi_c + \lambda_2 \phi_c^p \partial_x \phi_c \tag{4.35}$$

Getting Positivity, Continued

$\eta = 0$

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$$Lf_\star = \lambda_1 \phi_c + \lambda_2 \phi_c^p \partial_x \phi_c \tag{4.35}$$

Zeroing out Multipliers

$$\begin{aligned} \langle \partial_x \phi_c, Lf_\star \rangle &= \lambda_1 \langle \phi_c, \partial_x \phi_c \rangle + \lambda_2 \langle \phi_c^p \partial_x \phi_c, \partial_x \phi_c \rangle \\ 0 = \langle L\partial_x \phi_c, f_\star \rangle &= \lambda_1 \cdot 0 + \lambda_2 \underbrace{\int \phi_c^p (\partial_x \phi_c)^2}_{>0} \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} \langle \partial_c \phi_c, Lf_\star \rangle &= \lambda_1 \langle \partial_c \phi_c, \phi_c \rangle \\ 0 = -\langle \phi_c, f_\star \rangle = \langle L\partial_c \phi_c, f_\star \rangle &= \lambda_1 \tfrac{d}{dc} \mathcal{N}[\phi_c] \end{aligned} \tag{4.37}$$

Getting Positivity, Continued

$\eta = 0$

- We conclude $Lf_\star = 0$

Getting Positivity, Continued

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- We conclude $Lf_\star = 0$
- $f_\star \propto \partial_x \phi_c$

Getting Positivity, Continued

$\eta = 0$

- We conclude $Lf_\star = 0$
- $f_\star \propto \partial_x \phi_c$
- But $f_\star \perp \phi_c^p \partial_x \phi_c$; **contradiction**
- So $\eta > 0$

H^1 Bound

- We have proven that for $\langle f, \phi_c \rangle = 0$, $\langle f, \phi_c^p \partial_x \phi_c \rangle = 0$,

$$\langle Lf, f \rangle \geqslant \eta \|f\|_{L^2}^2$$

H^1 Bound

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$$\langle Lf, f \rangle \geqslant \eta \|f\|_{L^2}^2$$

- Want a lower bound in terms of H^1

$$\begin{aligned}\langle Lf, f \rangle &\geqslant \|\partial_x f\|_{L^2}^2 - K\|f\|_{L^2}^2 \\ &\geqslant \|\partial_x f\|_{L^2}^2 - \tfrac{K}{\eta}\eta\|f\|_{L^2}^2 \\ &\geqslant \|\partial_x f\|_{L^2}^2 - \tfrac{K}{\eta}\langle Lf, f \rangle\end{aligned}$$

Hence,

$$\langle Lf, f \rangle \geqslant \frac{1}{1 + K\eta^{-1}} \|\partial_x f\|_{L^2}^2 \Rightarrow \langle Lf, f \rangle \gtrsim \|f\|_{H^1}^2$$

Working with Near Orthogonality Conditions

- Our positivity is for $\langle f, \phi_c \rangle = 0$,$\left\langle f, \phi_c^p \partial_x \phi_c \right\rangle = 0$
- We have $\langle v, \phi_c \rangle = -\frac{1}{2}\|v\|_{L^2}^2$, $\left\langle v, \phi_c^p \partial_x \phi_c \right\rangle = 0$

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For v satisfying the above conditions:

$$\langle Lv, v \rangle \geqslant C_2 \|v\|_{H^1}^2 - C_3 \|v\|_{H^1}^3 - C_4 \|v\|_{H^1}^4$$

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For v satisfying the above conditions:

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Proof: Decompose v:

$$v = v_\perp + v_\| \tag{4.38}$$

where $v_\| = \langle v, \phi_c \rangle \phi_c$

Working with Near Orthogonality Conditions, Continued

Substituting into the quadratic form:

$$\left\langle Lv, v\right\rangle = \left\langle Lv_\perp, v_\perp\right\rangle + 2\left\langle Lv_\perp, v_\parallel\right\rangle + \left\langle Lv_\parallel, v_\parallel\right\rangle$$

and $v_\perp$ satisfies the assumptions, so

$$\begin{aligned}\left\langle Lv_\perp, v_\perp\right\rangle \gtrsim \|v_\perp\|_{H^1}^2 &\geqslant \|v\|_{H^1}^2 - 2\|v\|_{H^1}\left|\left\langle v, \phi_c\right\rangle\right| - \left|\left\langle v, \phi_c\right\rangle\right|^2\\ &\gtrsim \|v\|_{H^1}^2 - \|v\|_{H^1}^3 - \|v\|_{H^1}^4\end{aligned}$$

using near orthogonality $\left|\left\langle v, \phi_c\right\rangle\right| = \frac{1}{2}\|v\|_{H^1}^2$

Notes on the Result

- We proved $\|v\|_{H^1} = \mathrm{d}_1(u, \phi_c)$ is bounded in terms of invariants; since $\mathrm{d}_1 \asymp \mathrm{d}_c$, we infer $\mathrm{d}_c(u, \phi_c) \lesssim \epsilon$, closing the proof

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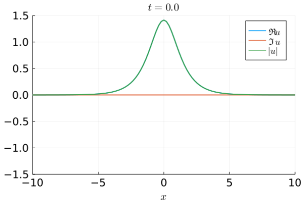
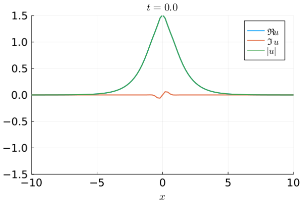
- Our proof relied on spectral information about L:
  - Kernel is just $\partial_x \phi_c$
  - L has single negative eigenvalue
- We did not require detailed/explicit information about ϕ_c or the ground state – this was the advance of Weinstein [23] over Benjamin/Bona [1, 2]

Necessity of the Sliding Metric in NLS





Sliding Metric and Orbits for NLS

- Must contend with both translations and phase shifts:

$$\mathrm{d}_\omega(f, g) = \inf_{y,\gamma} \sqrt{\|\nabla f(\cdot + y)e^{i\gamma} - \nabla g\|_{L^2}^2 + \omega \|f(\cdot + y)e^{i\gamma} - g\|_{L^2}^2} \tag{5.1}$$

- R_ω will orbitally stable if:

$$\mathrm{d}_\omega(\phi_0, R_\omega) \leqslant \delta \Rightarrow \mathrm{d}_\omega(\phi(t), R_\omega) \leqslant \epsilon \tag{5.2}$$

Orbital Stability for NLS

Theorem

For $\sigma d < 2$ and all $\omega > 0$, the NLS solitary wave is orbitally stable. For all $\epsilon > 0$, there exists $\delta > 0$, such that

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Strategy of Proof

- Decompose ϕ into R and a perturbation
- Taylor expand an action functional
- Prove that two quadratic forms are non-negative

Decomposition

- At optimal choice displacement and phase in the orbit (minimizing $d_\omega(\phi(t), R_\omega)$:

$$e^{i\gamma(t)}\phi(x + x_0(t), t) = R + w = R + u + iv \tag{5.3}$$

  where u and v are real valued
- At the optimal choice, we obtain $d + 1$ orthogonality conditions (useful later on):

$$\left\langle R_\omega^{2\sigma}\partial_{x_j} R_\omega, u\right\rangle = 0, \quad j = 1, \ldots, d \tag{5.4}$$

$$\left\langle R^{2\sigma+1}, v\right\rangle = 0 \tag{5.5}$$

Action Expansion

- Define the action:

$$\mathcal{S}_\omega[\phi] = \mathcal{H}[\phi] + \omega\mathcal{N}[\phi] \tag{5.6}$$

- Since $e^{i\gamma}\phi(\cdot + x_0) = R_\omega + w$,

$$\mathcal{S}_\omega[\phi] = \mathcal{S}_\omega[R_\omega + w]$$

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$$\begin{aligned}\Delta_\omega \mathcal{S}(t) &= \mathcal{H}[R_\omega + w(t)] - \mathcal{H}[R_\omega] + \omega(\mathcal{N}[R_\omega + w(t)] - \mathcal{N}[R_\omega]) \\ &= \langle L_+ u, u\rangle + \langle L_- v, v\rangle + r_{\omega,\sigma}[w]\end{aligned}$$

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- The linear operators are:

$$L_+ = -\nabla^2 + \omega - (2\sigma + 1)R_\omega^{2\sigma} \tag{5.7}$$

$$L_- = -\nabla^2 + \omega - R_\omega^{2\sigma} \tag{5.8}$$

- Remainder term, $r_{\omega,\sigma}[w] = \mathrm{O}(\|w\|_{H^1}^{2+\theta})$ with $\theta > 0$

Invariant Bound Strategy

- Suppose we prove:

$$
\begin{aligned}
\|u\|_{H^1}^2 - \|w\|_{H^1}^3 - \|w\|_{H^1}^4 &\lesssim \langle L_+ u, u\rangle \lesssim \|u\|_{H^1}^2\\
\|v\|_{H^1}^2 - \|w\|_{H^1}^3 - \|w\|_{H^1}^4 &\lesssim \langle L_- v, v\rangle \lesssim \|v\|_{H^1}^2
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- Then, as before:

$$\|w\|_{H^1}^2 - \|w\|_{H^1}^3 - \|w\|_{H^1}^4 - \|w\|_{H^1}^{2+\theta} \lesssim \Delta\mathcal{S}_\omega(0) \lesssim \|w\|_{H^1}^2 + \|w\|_{H^1}^{2+\theta}$$

and we obtain orbital stability

Linear Operators

- Recall the solitary wave equation:

$$-\nabla^2 R_\omega + \omega R_\Omega + R_\omega^{2\sigma+1} = 0$$

- We get that

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- For $d \geqslant 2$ – more challenging to show this is all that is in the kernel, assume true – proven for $d = 1, 3$ in [22, 23], a general result available in Kwong (89), [13], and also Chang et al. (07), [5]
- L_+ also has ground state ψ_0 with negative eigenvalue λ_0

Constraining the Bad Directions

Bad Directions

- L_- has a one dimensional null space – one bad direction
- L_+ has a d-dimensional null space and a negative eigenvalue – $d+1$ bad directions

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Constraints

- From orbit minimization, have $d+1$ constraints:

$$\left\langle R^{2\sigma+1}, v \right\rangle = \left\langle R_\omega^{2\sigma} \nabla R_\omega, u \right\rangle = 0 \tag{5.11}$$

- Need one more constraint – $\mathcal{N}[R+w] = \mathcal{N}[R]$ leads to **near orthogonality**

$$\langle R_\omega, u \rangle = -\tfrac{1}{2}\|w\|_{L^2}^2 \tag{5.12}$$

Positivity of L_-

Proposition

$$\alpha_- \equiv \inf_{\left\langle f, R_\omega^{2\sigma+1}\right\rangle = 0, \|f\|_{L^2}=1} \left\langle L_- f, f\right\rangle > 0 \tag{5.13}$$

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$$0 = \langle L_- R_\omega, f_\star\rangle = \langle R_\omega, L_- f_\star\rangle = \beta \int R_\omega^{2\sigma+2} \Rightarrow \beta = 0$$

Then $L_- f_\star = 0$, and $f_\star \propto R_\omega$; but $\langle R_\omega, R_\omega^{2\sigma+1}\rangle \neq 0$; contradiction.

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Also, $\langle L_- f, f\rangle \gtrsim \|f\|_{H^1}^2$ if we satisfy the orthogonality condition

Non-negativity of L_+

Proposition

If $\frac{d}{d\omega}\mathcal{N}[R_\omega] > 0$ *and* $\ker(L_+) = \{\nabla R_\omega\}$ *and there is only one negative eigenvalue, then*

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If $f \perp R_\omega^{2\sigma}\nabla R_\omega$ too, we obtain positivity, and the proof is completed in the same way as KdV.

VK Condition

- Using the scaling:

$$\|R_\omega\|_{L^2}^2 = \omega^{\frac{2-\sigma d}{2\sigma}} \|R_1\|_{L^2}^2 \tag{5.14}$$

- Increasing for $\sigma < 2/d$; orbitally stable

Negative Eigenvalue Count

- **CLAIM:**

$$\left\langle D^2 J[R] f, f \right\rangle = a_0 \langle L_+ f, f \rangle + a_1 \langle \varphi \otimes \chi f, f \rangle - a_2 \langle R \otimes R f, f \rangle \quad (5.15)$$

with $a_i > 0$ for all f

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- **Contradiction:**

$$\langle L_+ (\alpha_0 \psi_0 + \alpha_1 \psi_1), \alpha_0 \psi_0 + \alpha_1 \psi_1 \rangle = \lambda_0 \alpha_0^2 + \lambda_1 \alpha_1^2 < 0 \quad (5.18)$$

Generalized Problems

- For more general NLS/GP:

$$i\partial_t \phi = -\nabla^2 \phi + V(x)\phi + f(|\phi|^2)\phi \tag{6.1}$$

the associated $L_\pm$ are:

$$L_+ = -\nabla^2 + V(x) + \omega - f(R_\omega^2) - 2R_\omega^2 f'(R_\omega^2) \tag{6.2}$$

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- Computation of spectra of discretized $L_\pm$ is an option

Generalized Framework

- A complementary methodology is due to Grillakis, Shatah, & Strauss (87, 90), [10, 11, 12, 20]
- GSS has two advantages:
  - It directly predicts instability of solitary waves
  - It permits us to address solitary wave type solutions with more than one parameter (i.e. c for gKdV and ω for NLS/GP)
  - Example: gDNLS

    $$i\phi_t + i|\phi|^{2\sigma}\phi_x + \phi_{xx} = 0 \tag{6.5}$$

    has a two parameter, (ω, c), family of solitary wave solutions – studied in Liu, Simpson & Sulem (13), [15]
- GSS still requires the equivalent spectral analysis of $L_\pm$;

$$\mathrm{n}(L) = \mathrm{p}(\partial_{p_j} Q_i(p))) \Rightarrow \text{Orbital Stability} \tag{6.6}$$

Asymptotic Stability

- The solitary wave would be asymptotically stable if, in an appropriate distance,

$$\mathrm{d}(\phi_0, R_\omega) \leqslant \delta \Rightarrow \lim_{t\to\infty} \mathrm{d}'(\phi(t), R_{\omega_\star}) = 0 \tag{6.7}$$

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- Decomposition into a finite dimensional system + infinite dimensional perturbation:

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