One-dimensional NLS equation: the Inverse Scattering Method

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Introduction

- The notion of *integrable systems* originates in the 17th, 18th century to find explicit solutions to some dynamical systems (Kepler's laws of planetary motion).
- In the 19th century, notion of *Liouville integrability for Hamiltonian systems* was introduced: If, in a Hamiltonian system with n degree of freedom, n independent Poisson commuting integrals are known, the flow generated by H can be integrated explicitely by quadrature.
- The modern theory of integrable systems:
  Discovery by Gardner, Greene, Kruskal and Miura (1967) of a method to solve the Korteweg-de Vries equation: Express its solution u in terms of the spectral and scattering data of the stationary Schrödinger operator $-\partial_{xx} + u(x,t)$.
- Extended to several other nonlinear dispersive PDEs (NLS, mKdV, Sine-Cordon, Nonlocal NLS, Discrete NLS, Toda lattice, ...) as well as 2d dispersive PDEs (Kadomtsev-Petviashvili, Davey-Stewartson).

- Extended to many domains of math/physics: integrable stochastic models (Random matrix theory), orthogonal polynomials, Painlevé equations, knot theory, algebraic geometry, near-integrable models...)
- Today, Inverse Scattering Method in the context on nonlinear waves. The example of 1d NLS.

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I. Introduction to the Inverse Scattering Transform (IST)

- Some nonlinear PDEs can be solved by the *IST method*. They are referred to as *integrable evolution equations*.
- A classical example: the Korteweg-de Vries (KdV)

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(x, 0) = u_0(x). \tag{1}$$

- In 1967, Gardner, Greene, Kruskal and Miura presented a method, to solve the initial value problem (1) assuming u_0 decays sufficiently fast as $|x| \to \infty$. They showed how the solution of KdV can be constructed from the initial condition u_0.
- They introduced the linear spectral problem:

$$\mathcal{L}v \equiv -\partial_{xx}v + uv = \lambda v \tag{2}$$

where u plays the role of a *potential* for Schrödinger operator $\mathcal{L}$.
- They also explained how *soliton solutions* to the KdV equation are related to the eigenvalues of this spectral problem.

I.1. The Inverse Scattering Transform

The IST method can be explained with the help of the diagram

$$\begin{array}{ccc} u_0 & \longrightarrow & \mathcal{R}(u_0) = S(\lambda,0) \\ \text{Cauchy problem}\big\downarrow & & \big\downarrow\text{time evolution} \\ u(t) & \underset{\mathcal{I}}{\longleftarrow} & S(\lambda,t) \end{array}$$

(i) Solve the direct scattering problem at $t = 0$; i.e. find the scattering data $S(\lambda, 0)$ associated to the operator $\mathcal{L}$ with the initial potential u_0. The eigenvalues of $\mathcal{L}$ and behavior of its eigenfunctions as $|x| \to \infty$ determine the *scattering data.*

(ii) Time evolution of scattering data : Simple linear evolution

(iii) If $\mathcal{R}$ and $\mathcal{I} = \mathcal{R}^{-1}$ are well understood, one can reconstruct $u(t)$ from $S(\lambda, t)$ for all t, as well as find its behavior as $|t| \to \infty$.

I.2. The KdV equation

Gardner, Greene, Kruskal and Miura (1967) introduced 2 linear differential equations

$$\mathcal{L}v = -\partial_{xx}v + uv = \lambda v \tag{3a}$$

$$v_t = \mathcal{A}v = (\gamma + u_x)v - (4\lambda + 2u)v_x \tag{3b}$$

(γ is a constant and λ is the spectral parameter).

Proposition. Eqs. (3a)-(3b) are compatible ($v_{xxt} = v_{txx}$), if $\lambda_t = 0$ and u satisfies KdV.

$$v_{txx} = [(\gamma + u_x)(\lambda - u) + u_{xxx} + 6uu_x]v - (4\lambda + 2u)(\lambda - u)v_x$$

$$v_{xxt} = [(\gamma + u_x)(\lambda - u) - u_t + \lambda_t]v - (\lambda - u)(4\lambda + 2u)v_x$$

The set of linear operators $(\mathcal{L}, \mathcal{A})$ is called a *Lax pair*.

I.3. The Lax Pairs

Lax (1968) proposed a general setting. Consider the 2 linear differential equations:

$$\mathcal{L}\varphi = \lambda\varphi \tag{4a}$$

$$\varphi_t = \mathcal{A}\varphi \tag{4b}$$

Assuming $\lambda_t = 0$, these equations are compatible if and only if

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{A}] = 0$$

$[\mathcal{L}, \mathcal{A}] = \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$

Proof: Take d/dt of (4a) and use (4b) :

$$\mathcal{L}_t\varphi + \mathcal{L}\varphi_t = \lambda_t\varphi + \lambda\varphi_t$$
$$\mathcal{L}_t\varphi + (\mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L})\varphi = \lambda_t\varphi.$$

In the case of KdV:

If $\lambda_t = 0$ and $\mathcal{L}_t + [\mathcal{L}, \mathcal{A}] = 0 \quad \Leftrightarrow u$ satisfies KdV.

Remark. How does the scattering problem $-v_{xx} + uv = \lambda v$ relate to KdV?

Korteveg-de Vries (KdV)

$$u_t - 6uu_x + u_{xxx} = 0$$

Modified Korteweg-de Vries (mKdV)

$$m_t - 6m^2 m_x + m_{xxx} = 0$$

Miura transform : If m is solution of mKdV, then $u = m^2 + m_x$ is solution of KdV. (Riccati equation).

Introduce v s.t. $m = -\frac{v_x}{v}$, (reminiscent of Hopf-Cole transform)

$$-v_{xx} + uv = 0. \tag{5}$$

KdV is Galilean invariant: If u solves KdV, then $\tilde{u}(x,t) = u(x - 6\lambda t, t) + \lambda$ also solves KdV. Eq (5) becomes

$$\mathcal{L}v = -v_{xx} + uv = \lambda v.$$

This is the spectral problem associated to KdV.

I.4. The Zakharov-Shabat spectral problem

The stationary Schrödinger equation

$$-\varphi_{xx} + u\varphi = \lambda\varphi$$

with $\lambda = k^2$, can be represented as a system of 2 first-order equations:

$$\begin{cases} \psi_x = -ik\psi + u\varphi \\ \varphi_x = \psi + ik\varphi \end{cases} \qquad \begin{pmatrix} \psi \\ \varphi \end{pmatrix}_x = \begin{pmatrix} -ik & u \\ 1 & ik \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$$

Zakharov and Shabat(1972) proposed a generalisation of this system

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad v_x = Lv = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v$$

Choose $r = \pm q^*$. This is the spectral problem associated to NLS

$$iq_t + q_{xx} \mp 2|q|^2 q = 0$$

I.5. The Lax pair for NLS equation

$$v_x = Lv \equiv \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v \tag{6}$$

$$v_t = Bv \equiv \begin{pmatrix} -2ik^2 - irq & kq + iq_x \\ kr - ir_x & 2ik^2 + iqr \end{pmatrix} v \tag{7}$$

Proposition. The compatibility condition ($v_{xt} = v_{tx}$) together with the condition that the scattering parameter k is independent of t, is equivalent to the statement that q, r satisfy:

$$iq_t + q_{xx} - 2rq^2 = 0\,; \quad -ir_t + r_{xx} - 2qr^2 = 0$$

Choosing $r = \pm q^*$, the equations reduce to focusing/defocusing NLS

$$q_t + q_{xx} \mp 2|q|^2 q = 0.$$

The system (6) is called *the scattering problem*.
The system (7) is the *time evolution evolution*.
The operators $\{L, B\}$ constitute the Lax pair for 1d NLS.

I.6. The AKNS system

More generally, Ablowitz-Kaup-Newell-Segur (1974) considered the system

$$v_x = Lv \ , \quad v_t = Tv$$

where v is N-dim vector-valued function and L, T are $N \times N$ matrix operators. The compatibility condition $v_{xt} = v_{tx}$ implies

$$L_t - T_x - [L, T] = 0 \tag{8}$$

Consider the pair of operators ($N = 2$)

$$L = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where k is a spectral parameter, and A, B, C, D are functions of r, q, their derivatives, and k. The compatibility condition (8) together with the isospectral condition $k_t = 0$, implies a set of conditions on A, B, C, D.

Writing A, B, C, D as polynomials in k, these equations are solvable if q, r satisfy an evolution equation. One can obtain various integrable PDEs associated with the same spectral problem $v_x = Lv$.

Several 1d integrable PDEs: KdV, mKdV, sine Gordon, sinh Gordon, NLS, coupled NLS, Benjamin-Ono, Nonlocal NLS ...

Outline

- I. Introduction to the Inverse Scattering Transform
- II. The direct scattering map for 1d cubic NLS.
- III. Time evolution of scattering data.
- IV. The inverse scattering map.
- V. Long time behavior of solutions

II. The direct scattering map for 1d cubic NLS.

$$\begin{cases} iq_t + q_{xx} \mp 2|q|^2 q = 0 \,, & x \in \mathbb{R} \\ q(x,0) = q_0(x) \end{cases}$$

(+) : focusing; (-) : defocusing.
We will assume q_0 tends to 0 fast as $x \to \infty$.

The associated spectral problem is

$$v_x = Lv = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v$$

k is the spectral parameter. $r = \pm q^*$ (focusing/defocusing NLS)

GOAL: Detailed study of the spectral problem.
Determine the scattering data associated to potentials q, r.

Content of this section

1. Continuous spectrum $\mathbb{R}$ and associated eigenfunctions:
   Jost functions (defined by their behaviour as $x \to \pm\infty$)

2. Their properties:
   - (i) Analyticity in upper/lower complex plane in k-variable.
   - (ii) Behaviour as $k \to \infty$.
   - (iii) Reflection coefficient $\rho(k)$.
   - (iv) In the *focusing* case: Discrete eigenvalues and norming constants $\{(k_j, c_j)\}_1^J$.
   - (v) Symmetry reductions (due to the fact that $r = \pm q^*$).

3. The scattering data : $S(k,0) = \{\rho; (k_j, c_j)_1^J\}$.
   The direct scattering map $\mathcal{R}$: $q_0 \to S(k,0)$.

II.1. The scattering problem

$$v_x = Lv = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v.$$

If $q = r = 0$, the system has solutions $\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}$.

When the potentials $q, r \to 0$ rapidly as $x \to \pm\infty$, L has continuous spectrum $\mathbb{R}$ and eigenfunctions defined by their boundary conditions (*Jost functions*):

$$\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad \bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \quad x \to -\infty$$

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \quad x \to +\infty$$

Functions with constant boundary conditions:

$$M(x,k) = e^{ikx}\phi(x,k), \quad \bar{M}(x,k) = e^{ikx}\bar{\phi}(x,k)$$
$$N(x,k) = e^{-ikx}\psi(x,k), \quad \bar{N}(x,k) = e^{ikx}\bar{\psi}(x,k)$$

II.2. Properties of Jost functions

(i) Analyticity in upper/lower complex plane in k-variable

$M, \bar{M}, N, \bar{N}$ satisfy *Volterra integral equations*.

$$M(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 0 & q(x') \\ r(x')e^{2ik(x-x')} & 0 \end{pmatrix} M(x',k)dx'$$

and similar formulae for $\bar{M}, N, \bar{N}$.

Proposition . *If* $q, r \in L^1(\mathbb{R})$, *then* $M(x,k), N(x,k)$ *are analytic functions of* k, $\mathrm{Im}\, k > 0$, *and continuous for* $\mathrm{Im}\, k \geqslant 0$, *while* $\bar{M}(x,k), \bar{N}(x,k)$ *are analytic functions of* k, $\mathrm{Im}\, k < 0$, *and continuous for* $\mathrm{Im}\, k \leqslant 0$. *They are unique in the space of continuous functions.*

(ii) Behaviour of Jost functions as $|k| \to \infty$

One can compute the asymptotic behaviour for large k of the Jost functions, using integration by parts, and assuming sufficient regularity of q, r.

$$M(x,k) = \begin{pmatrix} 1 - \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx' \\ -\frac{1}{2ik}r(x) \end{pmatrix} + O(|k|^{-2})$$

$$\bar{N}(x,k) = \begin{pmatrix} 1 + \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx' \\ -\frac{1}{2ik}r(x) \end{pmatrix} + O(|k|^{-2})$$

$$N(x,k) = \begin{pmatrix} \frac{1}{2ik}q(x) \\ 1 - \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx' \end{pmatrix} + O(|k|^{-2})$$

$$\bar{M}(x,k) = \begin{pmatrix} \frac{1}{2ik}q(x) \\ 1 + \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx' \end{pmatrix} + O(|k|^{-2})$$

(iii) Scattering data: The reflection coefficient

The functions $\phi = (\phi^{(1)}, \phi^{(2)})^t$ and $\bar{\phi} = (\bar{\phi}^{(1)}, \bar{\phi}^{(2)})^t$ are *linearly independent*. Their Wronskian $W(\phi, \bar{\phi}) = \phi^{(1)}\bar{\phi}^{(2)} - \phi^{(2)}\bar{\phi}^{(1)}$ is independent of x, so we can compute it as $x \to -\infty$.

$$W(\phi, \bar{\phi}) = \lim_{x \to -\infty} W(\phi, \bar{\phi}) = 1$$

Similarly,

$$W(\psi, \bar{\psi}) = \lim_{x \to +\infty} W(\psi, \bar{\psi}) = -1$$

We can thus write ϕ and $\bar{\phi}$ as linear combinations of ψ and $\bar{\psi}$:

$$\phi(x,k) = b(k)\psi(x,k) + a(k)\bar{\psi}(x,k)$$
$$\bar{\phi}(x,k) = \bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k)$$

Comparing $W(\phi, \bar{\phi})$ at $\pm\infty$, we have:

$$a(k)\bar{a}(k) - b(k)\bar{b}(k) = 1.$$

We also have:

$$a(k) = W(\phi, \psi) = W(M, N); \quad \bar{a}(k) = -W(\bar{\phi}, \bar{\psi}) = W(\bar{M}, \bar{N})$$
$$b(k) = -W(\phi, \bar{\psi}); \qquad \bar{b}(k) = W(\bar{\phi}, \psi)$$

Proposition 1. *From the analyticity properties of $M, N, \bar{M}, \bar{N}$, we deduce that $a(k)$ is analytic in the upper k-plane and $\bar{a}(k)$ is analytic in the lower k-plane. In general, $b(k)$ and $\bar{b}(k)$ cannot be extended off the real k-axis.*

Proposition 2. *From the asymptotics of $M, N, \bar{M}, \bar{N}$, we have that*

$$a(k) \to 1 \text{ as } k \to \infty, \quad \text{Im } k > 0$$

$$\bar{a}(k) \to 1 \text{ as } k \to \infty, \quad \text{Im } k < 0.$$

We define the reflection coefficients

$$\rho(k) = b(k)/a(k); \quad \bar{\rho}(k) = \bar{b}(k)/\bar{a}(k) \ \text{ for } k \in \mathbb{R}.$$

(iv) Scattering data : Eigenvalues and norming constants

A proper eigenvalue of the original spectral problem is a complex value of k (Im k $\neq 0$) that corresponds to a solution v that is bounded as $x \to \pm\infty$.

Suppose that $a(k_j) = 0$, for some $k_j = \xi_j + i\eta_j$, $\eta_j > 0$. Then $\phi_j(x) \equiv \phi(x, k_j)$ and $\psi_j(x) \equiv \psi(x, k_j)$ are linearly dependent and there exists a complex constant c_j such that $\phi_j(x) = c_j \psi_j(x)$.

$$\phi_j(x) \ \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i(\xi_j + i\eta_j)x} \quad \text{as} \ \ x \to -\infty$$

$$\phi_j(x) = c_j \psi_j(x) \ \sim c_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i(\xi_j + i\eta_j)x} \quad \text{as} \ \ x \to +\infty$$

ϕ decays exponentially as $x \to \pm\infty$. Therefore k_j is an eigenvalue.

Similarly, the eigenvalues in Im k < 0 are the zeros of $\bar{a}$ and the zeros $\bar{k}_j = \bar{\xi}_j + i\bar{\eta}_j$ are such that $\bar{\phi}_j(x) = \bar{c}_j \bar{\psi}_j(x)$. The coefficients $\{c_j\}_1^J$ and $\{\bar{c}_j\}_1^J$ are called *norming constants.* Equivalently, the norming constants are defined by

$$M_j(x) = e^{2ik_j x} c_j N_j(x), \quad \bar{M}_j(x) = e^{-2ik_j x} \bar{c}_j \bar{N}_j(x).$$

We will assume the zeros are simple, no zeros on the real axis (no spectral singularities).

(v) Symmetry reductions

In the case of focusing/defocusing NLS, $r = \pm q^*$ and this implies various symmetries between the Jost functions, and consequently for the scattering data.

$$\bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,k^*) \\ N^{(1)}(x,k^*) \end{pmatrix}^* \qquad \bar{M}(x,k) = \begin{pmatrix} \mp M^{(2)}(x,k^*) \\ M^{(1)}(x,k^*) \end{pmatrix}^*$$

$$\bar{a}(k) = a^*(k^*), \;\; \text{Im}\, k < 0, \qquad \bar{b}(k) = \pm b(k) \;\; k \text{ real}$$

The eigenvalues come in conjugate pairs. ($\bar{k}_j = k_j^*, \bar{c}_j = \mp c_j^*$)

$$\bar{\rho}(k) = \mp \rho^*(k) \;\; k \text{ real}$$

Defocusing NLS. $r = q^*$. From the relation $a(k)\bar{a}(k) - b(k)\bar{b}(k) = 1$,

$$|a(k)|^2 - |b(k)|^2 = 1, \;\; k \text{ real}$$

$\Rightarrow$ $|a(k)| > 1$ for all k real. Also, the scattering problem is self-adjoint. Thus all eigenvalues must be real. $a(k)$ has no zeros in the complex plane. *No proper eigenvalues to the scattering problem.*

II.3. The scattering data and the direct map $\mathcal{R}$

- Defocusing NLS

$$q \;\to\; \mathcal{R}(q) = \rho$$

ρ is the reflection coefficient.

- Focusing NLS

$$q \;\to\; \mathcal{R}(q) = \{\, \rho,\; (k_j, c_j)_{j=1}^{J} \,\}$$

The $(k_j, c_j)\}$ are the eigenvalues and norming constants. Here $\mathrm{Im}\ \mathrm{k_j} > 0$. We will assume that all k_j are simple.

The properties of scattering coefficients are "similar" to those of the Fourier transform. Given an initial condition q_0 in the weighted Sobolev space

$$H^{1,1}(\mathbb{R}) = \{f : f, \partial_x f, |x| f \in L^2(\mathbb{R})\}$$

Theorem: The map $q_0 \to \rho_0$ is a map from $H^{1,1}(\mathbb{R})$ to $H^{1,1}(\mathbb{R})$. (Deift-Zhou, 2003)

III. Time evolution

Content of this section

1. Linear evolution of scattering data

$$a(k,t) = a(k,0), \quad b(k,t) = e^{-4ik^2 t}\, b(k,0)$$

– Evolution of the reflection coefficients $\rho(k,t) = b(k,t)/a(k,t)$

$$\rho(k,t) = \rho(k,0)\, e^{-4ik^2 t}$$

– The eigenvalues $k_j, \bar{k}_j$ (i.e. the zeros of a and $\bar{a}$) are constant.
Their location and their number are fixed.
– The norming constants evolve as

$$c_j(t) = e^{-4ik^2 t}\, c_j(0) \qquad \bar{c}_j(t) = e^{4ik^2 t}\, \bar{c}_j(0).$$

2. Infinite number of conservation laws for solutions of NLS

III.1. Time evolution of scattering data

We return to the 2nd equation (time-evolution) in the Lax pair:

$$v_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v = \begin{pmatrix} -2ik^2 - irq & kq + iq_x \\ kr - ir_x & 2ik^2 + iqr \end{pmatrix} v$$

Since q, r tend to 0 as $x \to \pm\infty$, we have that the time-dependent eigenfunctions satisfy ($A_\infty = -2ik^2$)

$$v_t = \begin{pmatrix} -2ik & 0 \\ 0 & 2ik \end{pmatrix} v \qquad \text{as } \; x \to \pm\infty \tag{9}$$

Solutions of (9) are linear combinations of $\begin{pmatrix} e^{-2ikt} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e^{2ikt} \end{pmatrix}$.

On the other hand,

$$\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \;\; \bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad x \to -\infty$$

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \;\; \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad x \to +\infty$$

Thus, joint solutions of the Lax pair must take the form:

$$\Phi(x,t;k) = e^{-2ikt}\phi, \quad \bar{\Phi}(x,t;k) = e^{2ikt}\bar{\phi},$$
$$\Psi(x,t;k) = e^{2ikt}\psi, \quad \bar{\Psi}(x,t;k) = e^{-2ikt}\bar{\psi}.$$

Time-evolution for ϕ:

$$\phi_t = \begin{pmatrix} A+2ik & B \\ C & -A+2ik \end{pmatrix}\phi; \quad \bar{\phi}_t = \begin{pmatrix} A-2ik & B \\ C & -A-2ik \end{pmatrix}\bar{\phi}\,.$$

As $x \to +\infty$, $\quad \phi_t \sim \begin{pmatrix} 0 & 0 \\ 0 & 4ik^2 \end{pmatrix}\phi; \quad \bar{\phi}_t \sim \begin{pmatrix} -4ik^2 & 0 \\ 0 & 0 \end{pmatrix}\bar{\phi}. \qquad (*)$

On the other hand, $\phi(x,k) = b(k)\psi(x,k) + a(k)\bar{\psi}(x,k)$ and

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \quad \text{as} \quad x \to +\infty.$$

Thus

$$\phi \sim \begin{pmatrix} a(k,t)\, e^{-ikx} \\ b(k,t)\, e^{ikx} \end{pmatrix} \quad \text{as} \quad x \to +\infty. \qquad (**)$$

Comparing (**) and (*)

$$a_t(k,t) = 0; \quad b_t(k,t) = 4ik^2 b(k,t).$$

Proposition 1. *Linear evolution of scattering data*

$$a(k,t) = a(k,0) \qquad \bar{a}(k,t) = a(k,0)$$
$$b(k,t) = e^{-4ik^2t}\, b(k,0) \qquad \bar{b}(k,t) = e^{4ik^2t}\, \bar{b}(k,0).$$

The evolution of the reflection coefficients $\rho(k,t) = b(k,t)/a(k,t)$ *and* $\bar{\rho}(k,t) = \bar{b}(k,t)/\bar{a}(k,t)$ *are given by*

$$\rho(k,t) = \rho(k,0)\, e^{-4ik^2t} \qquad \bar{\rho}(k,t) = \bar{\rho}(k,0)\, e^{4ik^2t}.$$

Proposition 2. *The eigenvalues* $k_j, \bar{k}_j$ *(i.e. the zeros of* a *and* $\bar{a}$*) are constant. Their location and their number are fixed.*

The norming constants evolve as

$$c_j(t) = e^{-4ik^2t}\, c_j(0) \qquad \bar{c}_j(t) = e^{4ik^2t}\, \bar{c}_j(0).$$

III.2. Infinite number of conservation laws

For simplicity, assume that $a(k), \bar{a}(k)$ have no zeros (NLS defocusing)

The functions $a(k)$ and $\bar{a}(k)$ are time-independent

(a) Asymptotic expansion of $\log a(k)$ for large k

- $a(k)$ analytic in $\mathbb{C}^+$, has no zeros in $\mathbb{C}^+$, $a(k) \to 1$ as $|k| \to \infty$
- $\bar{a}(k)$ analytic in $\mathbb{C}^-$, has no zeros in $\mathbb{C}^-$, $\bar{a}(k) \to 1$ as $|k| \to \infty$

$$\log a(k) = \sum_{n=0}^{\infty} \frac{\Gamma_n}{(2ik)^n}, \quad \Gamma_n = -\frac{1}{\pi} \int_{\mathbb{R}} (2is)^n \log(1 - |\rho(s)|^2) ds$$

(b) Relation between $a(k)$ and Jost functions at $x = \pm\infty$

$$a(k) \sim M_1 \quad \text{as } x \to +\infty.$$

Writing $M = e^{\sigma}$ and using that M satisfies the spectral problem, we get a Riccati equation for $\gamma = \partial_x \sigma$.

(c) Matching large k-asymptotics for $\log a$ and γ.

(e) Infinite number of conservation laws expressed both in terms of NLS solution and moments of ρ.

(a) Asymptotic expansion of $\log a(k)$ for large k.

By Cauchy integral theorem,

$$\log a(k) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log a(s)}{s-k}\,ds, \quad 0 = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log \bar{a}(s)}{s-k}\,ds, \;\; \mathrm{Im\,k} > 0$$

$$\log a(k) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log a(s)\bar{a}(s)}{s-k}\,ds, \;\; \mathrm{Im\,k} > 0$$

Defocusing NLS: $r = q^*$, $a(s)\bar{a}(s) = (1-|\rho(s)|^2)^{-1}$.

One can recover $a(k), \bar{a}(k)$ from the reflection coefficient ρ.

$$\log a(k) = \sum_{n=0}^{\infty} \frac{\Gamma_n}{(2ik)^n}, \;\; \Gamma_n = -\frac{1}{\pi}\int_{\mathbb{R}} (2is)^n \log(1-|\rho(s)|^2)ds$$

(b) Relation between $a(k)$ and Jost functions.

The scattering problem $v_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v$ has solutions ϕ, ψ

$$\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \ \ \bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \ \ x \to -\infty$$

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \ \ \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \ \ x \to +\infty$$

$$\phi(x,k) = b(k)\psi(x,k) + a(k)\bar{\psi}(x,k)$$

$$M(x,k) = e^{ikx}\phi(x,k)$$

Lemma . *As* $x \to +\infty$, $\phi_1(x,k) \sim a(k)e^{-ikx}$, thus

$$a(k) \sim M_1 \ \text{ as x} \to +\infty$$

M satisfies the scattering problem

$$\begin{cases} \partial_x M_1 = qM_2 \,, \\ \partial_x M_2 = 2ikM_2 + M_1 \,, \end{cases}$$

Eliminating M_2

$$\partial_{xx} M_1 = \big(\frac{q_x}{q} + 2ik\big)\partial_x M_1 + qrM_1$$

Write $M_1 = e^{\sigma}$, we have $\partial_x M_1 = \sigma_x e^{\sigma}, \;\; \partial_{xx} M_1 = (\sigma_{xx} + \sigma_x^2)e^{\sigma}.$

Lemma . $\gamma = \partial_x \sigma$ *satisfies the Riccati equation*

$$\gamma^2 + \gamma_x = \frac{q_x}{q} + 2ik\gamma + qr.$$

Proof: Write equation for M_1 in terms of γ and substitute.

Rewrite the Riccati equation as

$$2ik\gamma = \gamma^2 - qr + q\Big(\frac{\gamma}{q}\Big)_x \qquad (*)$$

As $|k| \to \infty$, $M_1 = e^{\sigma} \to 1$, thus σ and $\gamma = \partial_x \sigma \to 0$.
We now write an *asymptotic expansion of* γ *in powers of* $1/k$

$$\gamma(x,k,t) = \sum_{n=1}^{\infty} \frac{\gamma_n(x,t)}{(2ik)^n}$$

Substituting in (*) and matching the powers of $1/2ik$, we have the recurrence formula

$$\gamma_1 = -qr, \quad \gamma_2 = q(\gamma_1/q)_x = -qr_x$$

$$\gamma_{n+1} = q\Big(\frac{\gamma_n}{q}\Big)_x + \sum_{j=1}^{n-1} \gamma_j \gamma_{n-j}.$$

(c) Matching asymptotic expansions

$$\sigma(x,k) \sim \log a(k), \quad \text{as } x \to +\infty.$$

also (from the behavior of ϕ at $x = -\infty$)

$$\sigma(x,k) \to 0, \quad \text{as } x \to -\infty.$$

$$\log a(k) \sim \sigma(x,k,t)|_{x=\infty} = \int_{-\infty}^{+\infty} \partial_x \sigma(x,k,t)dx = \int_{-\infty}^{+\infty} \gamma(x,k,t)dx$$

We now identify the asymptotic expansion in k

$$\log a(k) = \sum_{n=0}^{\infty} \frac{\Gamma_n}{(2ik)^n} = \sum_{n=1}^{\infty} \frac{\int_{-\infty}^{\infty} \gamma_n(x,t)dx}{(2ik)^n}$$

$$\Gamma_n = \int_{-\infty}^{\infty} \gamma_n(x,t)dx$$

(d) $a(k)$ is independent of time, thus also all Γ_n, leading to an infinite number of conservation laws:

$$\Gamma_n = \int_{-\infty}^{\infty} \gamma_n(x,t)dx = \text{Const.}$$

The first three invariants (mass, momentum, Hamiltonian)

$$\Gamma_1 = \int_{\mathbb{R}} |q|^2 \, dx$$

$$\Gamma_2 = \int_{\mathbb{R}} q_x^* q \, dx$$

$$\Gamma_3 = \int_{\mathbb{R}} (\mp |q_x|^2 + |q|^4) \, dx$$

Note that the Γ_n are also expressed in terms of the scattering data in the form: (see page 27)

$$\Gamma_n = -\frac{1}{\pi} \int_{\mathbb{R}} (2is)^n \log(1 - |\rho(s)|^2) ds$$

Remark. In the case where $a(k), \bar{a}(k)$ have zeros $\{(k_j, \bar{k}_j)\}_{j=1}^{J}$, (focusing NLS), introduce

$$\alpha(k) = \prod_{m=1}^{J} \tfrac{k-k_m^*}{k-k_m} a(k), \quad \bar{\alpha}(k) = \prod_{m=1}^{J} \tfrac{k-k_m}{k-k_m^*}\, \bar{a}(k)$$

$\alpha(k)$ analytic in $\mathbb{C}^+$, has no zeros ; $\bar{\alpha}(k)$ analytic in $\mathbb{C}^-$, has no zeros.

$$\log \alpha(k) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \alpha(s)}{s-k}\, ds, \quad \mathrm{Im\,k} > 0$$

$$0 = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \bar{\alpha}(s)}{s-k}\, ds, \quad \mathrm{Im\,k} > 0$$

$$\log a(k) = \sum_{m=1}^{J} \log \frac{k-k_m^*}{k-k_m} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log a(s)\bar{a}(s)}{s-k}\, ds, \quad \mathrm{Im\,k} > 0$$

Use $a(s)\bar{a}(s) = (1 + |\rho(s)|^2)^{-1}$, and write $\log a(k) = \sum_{n=0}^{\infty} \frac{\Gamma_n}{(2ik)^n}$

$$\Gamma_n = \sum_{1}^{J} \frac{(2ik_m^*)^n - (2ik_m)^n}{n} - \frac{1}{\pi} \int_{\mathbb{R}} (2is)^n \log(1 - |\rho(s)|^2) ds.$$

IV. The inverse scattering map

The inverse problem consists in reconstructing the solution q of NLS from the scattering data $\{ \rho, \ (k_j, c_j)_{j=1}^{J} \}$.

- The original method (Zakharov-Shabat 1972) uses the *Gelfand-Levitan-Marchenko integral equation*, introduced by Gardner, Green, Kruskal, Miura (1967) for KdV.
- Another approach uses *Riemann-Hilbert problems* (Manakov 1972, Its 1982, Beals-Coifman 1984, Deift-Zhou 1993). It is well adapted to the study of long-time behaviour of solutions.

A *Riemann-Hilbert problem (RHP)* refers to the problem of finding a piecewise analytic function knowing its jump along a given contour, and an additional condition at ∞.
(More generally sectionally meromorphic functions, with additional residue conditions).

Content of this section

1. Preliminaries: Cauchy operators; example of a RHP

2. From reflection coefficient to solution of NLS. Statement of RHP. Statement of result.

3. Construction of inverse scattering map

IV.1. Preliminaries

(a) Cauchy operators. If $f \in L^2(\mathbb{R})$, the Cauchy integral

$$Cf(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)}{s-z}ds$$

defines a function bounded and analytic in $\mathbb{C}\backslash\mathbb{R}$, with $Cf(z) \to 0$ as $z \to \infty$. The nontangential limits (Cauchy projectors)

$$C^{\pm}f(k) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(s)}{s-(k \pm i0)}ds$$

exist and satisfy the *Plemelj-Sokhotski formula*

$$C^+f - C^-f = f\ , \quad C^+f + C^-f = -\mathcal{H}f,$$

where $\mathcal{H}f(k) = \lim_{\epsilon\to 0} \frac{1}{\pi i}\int_{|k-s|>\epsilon} \frac{f(s)}{k-s}ds$ is the Hilbert transform.

(b) A model scalar RHP. Let $\rho \in H^1$, $\|\rho\|_{L^\infty} < 1$, and $\xi \in \mathbb{R}$ fixed. Find a function $\delta(z)$ that is analytic for $z \in \mathbb{C} \backslash (-\infty, \xi]$ and satisfies the following conditions

1. $\delta(z) \to 1$ as $z \to \infty$,
2. $\delta(k)$ has continuous boundary values $\delta_\pm(k) = \lim_{\epsilon \downarrow 0} \delta(k \pm i\epsilon)$ for $k \in (-\infty, \xi)$,
3. $\delta_\pm$ obey the jump relation

$$\delta_+(k) = \begin{cases} \delta_-(k)\left(1 - |\rho(k)|^2\right), & k \in (-\infty, \xi) \\ \delta_-(k), & z \in (\xi, \infty) \end{cases}$$

This RHP has a unique solution given by

$$\delta(z) = \exp\left(i \int_{-\infty}^{\xi} \frac{1}{s - z} \kappa(s)\, ds\right), \quad \kappa(s) = -\frac{1}{2\pi} \log\left(1 - |\rho(s)|^2\right)$$

IV.2. The case of no eigenvalues – From the reflection coefficient to potential q

In the case where there are no eigenvalues, i.e. no zeros of a (defocusing NLS), the scattering data are reduced to the reflection coefficient $\rho(k)$.

- *Beals and Coifman (1984) identified solutions of the spectral problem that have piecewise continuation to $\mathbb{C}\backslash\mathbb{R}$ and solve a Riemann-Hilbert Problem (RHP) completely determined by ρ. These are called Beals-Coifman solutions.*
- From the large k behaviour of the solution to the RHP, we will recover q.
- This method defines the *inverse scattering map* :

$$\mathcal{I} : \rho \to q$$

Theorem: The map $\rho \to q$ is a map from $H^{1,1}(\mathbb{R})$ to $H^{1,1}(\mathbb{R})$. (Deift-Zhou, 2003)

Riemann-Hilbert Problem

Let $\rho \in H^{1,1}(\mathbb{R})$, $\|\rho\|_{L^\infty} < 1$, x fixed, find a 2×2 matrix $\mathbf{m}(x; z)$ s.t.

1. $\mathbf{m}(x; z)$ analytic in $\mathbb{C}\backslash\mathbb{R}$, with continuous boundary values $\mathbf{m}_\pm(x; k) = \lim_{\epsilon\to 0^+} \mathbf{m}_\pm(x; k + i\epsilon)$
2. $\mathbf{m}(x; z) \to \mathbb{I}$ as $|z| \to \infty$
3. The jump relation along the contour $\mathbb{R}$ is $\mathbf{m}_+(x; k) = \mathbf{m}_-(x; k)\mathbf{V}(k)$

$$\mathbf{V}(k) = \begin{pmatrix} 1 - |\rho(k)|^2 & -\bar{\rho}(k)e^{-2ikx} \\ \rho(k)e^{2ikx} & 1 \end{pmatrix}$$

Proposition 1. *Given $\rho \in H^{1,1}(\mathbb{R})$, $\|\rho\|_{L^\infty} < 1$, there exists a unique matrix-solution $\mathbf{m}(x; z)$ to the above RHP.*

Proposition 2. From the solution $\mathbf{m}(x; z)$, one obtains

$$q(x) = \lim_{z\to\infty} 2iz\, \mathbf{m}_{12}(x; z)$$

where the limit is taken as $|z| \to \infty$ in any proper subsector of the upper or lower half-plane.

IV.3. Construction of the inverse scattering map

- (a) Where does this RHP come from? (next 2 slides)
- (b) Construction of a solution $\mathbf{m}(x; z)$ to the RHP.
  Existence and uniqueness. (next 2 slides)
- (c) Comparing the large z behaviour of $\mathbf{m}(x; z)$ to that of the Jost functions, one finds $q \sim 2iz\, \mathbf{m}_{12}(x, z)$.

(a) Beals-Coifman solutions and formulation of a RHP

- $M(x,k), N(x,k)$ are analytic functions of k, $\operatorname{Im} k > 0$, continuous for $\operatorname{Im} k \geqslant 0$,
- $\bar{M}(x,k), \bar{N}(x,k)$ are analytic functions of k, $\operatorname{Im} k < 0$, continuous for $\operatorname{Im} k \leqslant 0$,
- $a(k)$ (resp. $\bar{a}(k)$) is analytic in $\operatorname{Im} k > 0$ (resp. $\operatorname{Im} k < 0$)
- $\mu(x,k) = \frac{M(x,k)}{a(k)}$ is analytic in $\operatorname{Im} k > 0$
- $\bar{\mu}(x,k) = \frac{\bar{M}(x,k)}{\bar{a}(k)}$ is analytic in $\operatorname{Im} k < 0$

Introduce 2×2 matrices:

$$\mathbf{m}^{(+)}(x,k) = (\mu(x,k), N(x,k)), \quad \mathbf{m}^{(-)}(x,k) = (\bar{N}(x,k), \bar{\mu}(x,k))$$

and the piecewise analytic function (*Beals-Coifman solution*)

$$\mathbf{m}(x,k) = \begin{cases} \mathbf{m}^{(+)}(x,k) & \operatorname{Im} k > 0 \\ \mathbf{m}^{(-)}(x,k) & \operatorname{Im} k < 0 \end{cases}$$

From the large k behavior of $M, N, \bar{M}, \bar{N}$, $\mathbf{m}(x,k) \to \mathbb{I}$ as $|k| \to \infty$.

$\mathbf{m}(x,k)$ is also normalized at $x \to +\infty$,

$$\mathbf{m}(x,k) \to \mathbb{I}, \quad \text{as } |x| \to \infty.$$

Using the relations $M = e^{ikx}\phi,\ N = e^{-ikx}\psi,\ \bar{M} = e^{-ikx}\bar{\phi},\ \bar{N} = e^{ikx}\bar{\psi}$, $\phi = b\psi + a\bar{\psi},\ \bar{\phi} = \bar{a}\psi + \bar{b}\bar{\psi}$, we have for k real,

$$\mu(x,k) = \bar{N}(x,k) + \rho(k)e^{2ikx}N(x,k)$$
$$\bar{\mu}(x,k) = N(x,k) + \bar{\rho}(k)e^{-2ikx}\bar{N}(x,k)$$

These can be seen as *jump conditions* for $\mathbf{m}_\pm(x,k) = \lim_{\epsilon\to 0}\mathbf{m}(x, k \pm i\epsilon)$ RHP satisfied by the 2×2 matrix $\mathbf{m}$:

1. $\mathbf{m}(x;z)$ analytic in $\mathbb{C}\setminus\mathbb{R}$ for each x, with continuous boundary values $\mathbf{m}_\pm(x;k) = \lim_{\epsilon\to 0^+}\mathbf{m}_\pm(x;k+i\epsilon)$
2. $$\mathbf{m}(x,k) \to \mathbb{I}, \text{ as } |k| \to \infty \tag{10}$$

$$\mathbf{m}_+(x,k) - \mathbf{m}_-(x,k) = \mathbf{m}_-(x,k)\mathbf{v}(x,k) \tag{11}$$
$$\mathbf{v}(x,k) = \begin{pmatrix} -\rho(k)\bar{\rho}(k) & -\bar{\rho}(k)e^{-2ikx} \\ \rho(k)e^{2ikx} & 0 \end{pmatrix}$$

(b) Construction of a solution $\mathbf{m}(x;z)$ to the RHP.

Factorisation of the jump matrix

$$\mathbf{V}(x,k) = \begin{pmatrix} 1-|\rho(k)|^2 & -\bar{\rho}(k)e^{-2ikx} \\ \rho(k)e^{2ikx} & 1 \end{pmatrix}$$

in the form: $\mathbf{V} = (\mathbb{I} - w_x^-(k))^{-1}(\mathbb{I} + w_x^+(k))$

$$w_x^+(k) = \begin{pmatrix} 0 & 0 \\ e^{2ikx}\rho(k) & 0 \end{pmatrix}, \quad w_x^-(k) = \begin{pmatrix} 0 & -e^{-2ikx}\bar{\rho}(k) \\ 0 & 0 \end{pmatrix}$$

Next introduce

$$\nu(x,k) = \mathbf{m}_+(x,k)(\mathbb{I} + w_x^+(k))^{-1} = \mathbf{m}_-(x,k)(\mathbb{I} - w_x^-(k))^{-1}$$

and check that

$$\nu(w_x^+ + w_x^-) = \mathbf{m}_+ - \mathbf{m}_-$$

Proposition. For $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\mathbf{m}(x,z) = \mathbb{I} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\nu(x,s)(w_x^+(s) + w_x^-(s))}{s-z}ds$$

Construction of ν.

We have:

$$\mathbf{m}_+ = \nu(\mathbb{I} + w_x^+) = \nu + \nu w_x^+$$

Using Cauchy projectors, $\mathbf{m}_+ = \mathbb{I} + C^+(\nu(w_x^+ + w_x^-))$. Thus,

$$\nu + \nu w_x^+ = \mathbb{I} + C^+(\nu(w_x^+ + w_x^-)). \tag{12}$$

Also

$$C^+(\nu w_x^+) - C^-(\nu w_x^+) = \nu w_x^+. \tag{13}$$

Adding (12) and (13)

$$\nu = \mathbb{I} + C^+(\nu w_x^-) + C^-(\nu w_x^+)$$

$$\nu = \mathbb{I} + C_w \nu \tag{14}$$

where $C_w h = C^+(h w_x^-) + C^-(h w_x^+)$. C_w is called the *Beals-Coifman integral operator* and (14) the *Beals-coifman integral equation*.

Proposition. Fix $x \in \mathbb{R}$. Assume $\rho \in H^{1,1}(\mathbb{R})$ with $\|\rho\|_L^\infty < 1$. There exists a unique solution ν to (14) with $\nu - \mathbb{I} \in H^1(\mathbb{R})$.

(c) Large k behavior

m identifies to the Beals-Coifman solutions defined earlier.

Tracing back the definitions,
$\mathbf{m}_{12}(x,k) \leftrightarrow N_1(x,k)$ if $\operatorname{Im} k > 0$,
$\mathbf{m}_{12}(x,k) \leftrightarrow \bar{M}_1(x,k)/a(k)$ if $\operatorname{Im} k < 0$,
As $k \to \infty$, we proved earlier:

$$N(x,k) = \begin{pmatrix} \frac{1}{2ik} q(x) \\ 1 - \frac{1}{2ik} \int_x^{+\infty} q(x')r(x')dx' \end{pmatrix} + O(|k|^{-2})$$

$$\bar{M}(x,k) = \begin{pmatrix} \frac{1}{2ik} q(x) \\ 1 + \frac{1}{2ik} \int_{-\infty}^{x} q(x')r(x')dx' \end{pmatrix} + O(|k|^{-2})$$

$$\begin{aligned} q(x) &= \lim_{k\to\infty} 2ik\, \mathbf{m}_{12}(x;k) \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \nu_{11}(x,\xi)\overline{\rho}(\xi) e^{-2ikx} d\xi. \end{aligned}$$

Remark: In this analysis, t plays the role of a parameter. To write the inverse scattering map for $t > 0$, we need to replace ρ by

$$\rho(k,t) = \rho_0(k)e^{-4ik^2 t}.$$

In the RHP, the exponential factor e^{-2ikx} is replaced by $e^{-2ikx-4i\xi^2 t}$
We will write

$$e^{-2ikx-4i\xi^2 t} = e^{-2it\theta}$$

where θ is the phase function: $\quad \theta(k;x,t) = 2k^2 + kx/t.$
$\partial_k\theta(k;x,t) = 4k + x/t$. Stationary phase point is $\xi = -x/4t$.

Large t behavior : Study of fast oscillatory integrals. Method of stationary phase/steepest descent. [Zakharov-Manakov 1976, Deift-Zhou 2003]

The inverse scattering map. Summary. NLS defocusing

Riemann-Hilbert Problem. Given ρ_0, x, t, find $\mathbf{m}(z; x, t)$ s.t.

1. $\mathbf{m}(x, k; t)$ analytic in $\mathbb{C} \backslash \mathbb{R}$ for each x, t, with continuous boundary values $\mathbf{m}_{\pm}(x, k; t)$ on $\mathbb{R}$.
2. $\mathbf{m}(z; x, t) \to \mathbb{I}$ as $|z| \to \infty$
3. The jump relation $\mathbf{m}_+(x, k; t) = \mathbf{m}_-(x, k; t) V(k)$

$$V(k) = \begin{pmatrix} 1 - |\rho_0(k)|^2 & -\rho_0(k) e^{-2it\theta} \\ \rho_0(k) e^{2it\theta} & 1 \end{pmatrix}$$

The real phase function θ is given by $\theta(k; x, t) = 2k^2 + \frac{x}{t} k$

From the solution $\mathbf{m}(z; x, t)$ of RHP above, one obtains $q(x, t)$

$$q(x, t) = \lim_{z \to \infty} 2iz\, \mathbf{m}_{12}(z; x, t)$$

Large t behavior : Study of fast oscillatory RHP.
[Zakharov-Manakov 1976, Deift-Zhou 2003]

The case of poles

- If scattering data : $\{\rho \equiv 0, (k_j, c_j)_{j=1}^N\} \to q =$ N-soliton
- In the special case $N = 1$, $(\xi + i\eta, c)$, the corresponding solution is the 1-soliton:

$$q_{sol}(x,t) = 2\eta \operatorname{sech}(2\eta(x + 2\xi t - x_0))e^{-2i(\xi x + (\xi^2 - \eta^2)t}e^{-i\phi_0}$$

where $x_0 = \frac{1}{2\eta} \log |c/2\eta|, \; \phi_0 = \pi/2 + \arg(\mathrm{c})$

$$|q|_{L^2}^2 = 4\eta$$

V. Large-time behavior of solutions of defocusing NLS

Content of this section

- (a) Statement of the long time behaviour result using IST method
- (b) Statement of the long time behaviour result using direct PDE methods
- (c) Summary of the steps of the analysis
- (d) Comments on each step of the analysis
- (e) Long time asymptotics and soliton resolution for focusing NLS.

V. Large-time behavior of solutions of defocusing NLS

Theorem. (Deift-Zhou 2003)

$$iq_t + q_{xx} - 2|q|^2 q = 0,\ q(x,0) = q_0(x)$$

Let $q_0 \in H^{1,1}(\mathbb{R})$. As $t \to \infty$,

$$q(x,t) \sim t^{-1/2}\alpha(\xi)e^{ix^2/4t - i\nu(\xi)\log(2t)} + O(t^{-(\frac{1}{2}+\kappa)}),$$

$0 < \kappa < 1/4$, $\xi = -x/4t$ (stationary phase point) , and α and ν given in terms of the reflection coefficient ρ_0 associated to initial condition q_0:

$$\nu(k) = -\frac{1}{2\pi}\log(1 - |\rho_0(k)|^2);\quad |\alpha(k)| = \sqrt{\frac{\nu(k)}{2}}$$

$$\arg\alpha(k) = \frac{1}{\pi}\int_{-\infty}^{k} \frac{\log(1 - |\rho_0(s)|^2)}{k - s}ds + \frac{\pi}{4} + \arg\Gamma(i\nu(k)) + \arg(\rho_0(k)).$$

Direct PDE methods : [Hayashi, Naumkin and Uchida (1999)]
1d cubic NLS with general nonlinearities with first-order derivatives.

$$iu_t + \partial_{xx}u = N(u, u^*, \partial_x u, \partial_x u^*)$$

Assuming smooth initial conditions, small in some weighted Sobolev spaces, they proved :

- Global existence of solutions
- There exist asymptotic states $u^{\pm} \in L^2 \cap L^{\infty}$ and real valued functions $g^{\pm} \in L^{\infty}$ such that

$$u(x,t) \sim \frac{1}{\sqrt{t}} u^{\pm}\left(\frac{x}{2t}\right) \exp\left(\frac{ix^2}{4t} \pm i g^{\pm}\left(\frac{x}{2t}\right) \log|t|\right) + O(|t|^{-1/2-\alpha})$$

uniformly in $x \in \mathbb{R}$, with $0 < \alpha < 1/4$.

V.1. Steps of the analysis to find large t behavior of solution of RHP

– Extension of the classical methods of stationary phase/steepest descent for oscillatory integrals.

– For large t, RHP reduces to a RHP with nontrivial jumps only in a small neighborhood of stationary phase point $\xi = -x/4t$.

– After further reductions, the RHP becomes a universal one, solvable in terms of special functions, solutions of *the parabolic cylinder equation*.

– Leading asymptotic behavior of q(x,t) as $t \to \infty$ from the reconstruction formula.

– Method of nonlinear steepest descent. Rigorous analysis and error estimates for oscillatory RHPs, minimal assumptions on initial data $q_0 \in H^{1,1}(\mathbb{R})$. (Deift-Zhou 1993, 2003).

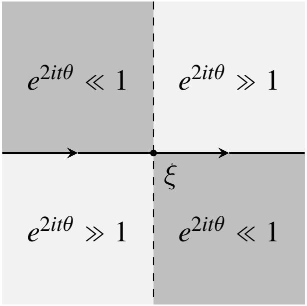
– Focusing NLS: Soliton resolution (Zakharov-Shabat 1972, Borghese-Jenkins-McLaughlin 2018).

Step 1: Preparation for steepest descent

Phase function : $\theta(\lambda) = \theta(\lambda; x, t) = 2\lambda^2 + \frac{x}{t}\lambda$

$\theta'(\lambda) = 4\lambda + x/t$. One stationary point $\xi = -x/4t$.

The regions of growth and decay of the exponential factor $e^{2it\theta}$ in the λ-plane, ($t > 0$).



Need to separate the factor $e^{it\theta}$ to $e^{-it\theta}$ algebraically.

This is done by writing a new RHP for

$$\mathbf{m}_1 = \mathbf{m} \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$$

$$\delta(z) = \exp\left(i \int_{-\infty}^{\xi} \frac{1}{s-z} \kappa(s)\, ds \right), \quad \kappa(s) = -\frac{1}{2\pi} \log\left(1 - |\rho(s)|^2\right)$$

analytic for $z \in \mathbb{C} \backslash (-\infty, \xi]$ satisfies the *model scalar RHP*

1. $\delta(z) \to 1$ as $z \to \infty$,
2. $\delta(z)$ has continuous boundary values $\delta_\pm(z) = \lim_{\downarrow 0} \delta(z \pm i)$ for $z \in (-\infty, \xi)$,
3. $\delta_\pm$ obey the jump relation (model scalar RHP)

$$\delta_+(z) = \begin{cases} \delta_-(z) \left(1 - |\rho(z)|^2\right), & z \in (-\infty, \xi) \\ \delta_-(z), & z \in (\xi, \infty) \end{cases}$$

The new RHP allows the contours to be deformed with the exponential factors $e^{\pm it\theta}$ having maximum decay in $(z - \xi)$.

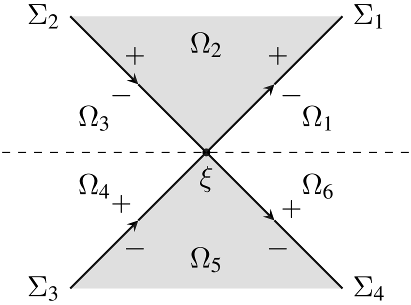
Step 2: Contour deformation from $\mathbb{R}$ to $\Sigma^{(2)}$.

Remove jumps on $\mathbb{R}$, introduce new jumps on augmented contour $\Sigma^{(2)}$

$$\mathbf{m}_2 = \mathbf{m}_1 \mathcal{R}$$

$\mathcal{R}$ piecewise continuous matrix.

Figure: new contour $\Sigma^{(2)}$



Step 3: "Freezing coefficients"

- Scattering data are replaced by their value at stationary point ξ.
- The new unknown $\mathbf{m}_2$ has jump matrices written in terms of the scaled variable $\zeta(z) = \sqrt{t}(z - \xi)$.
  Phase $e^{2it\theta} = e^{-i\zeta^2/2} e^{ix^2/4t}$. The factor $e^{-i\zeta^2/2}$ will be important in the identification of parabolic cylinder functions.
- This transformation introduces some non-analyticity for $\mathbf{m}_2$ in regions $\Omega_1, \Omega_3, \Omega_4, \Omega_6$. The new unknown $\mathbf{m}_2$ satisfies a mixed $\bar{\partial}$-RHP problem.
- $\mathbf{m}_2 = \mathbf{m}_3 \mathbf{m}^{\mathrm{PC}}$ with control on $\mathbf{m}_3$ for t large:

$$\mathbf{m}_3(z; x, t) = \mathbb{I} + \frac{1}{z} \mathbf{m}_3^{(1)}(x, t) + o\left(\frac{1}{z}\right)$$

$$\left|\mathbf{m}_3^{(1)}(x, t)\right| \lesssim t^{-3/4}.$$

- RHP for $\mathbf{m}^{\mathrm{PC}}$ with jumps along contour $\Sigma^{(2)}$

Step 4: Solving RHP for $\mathbf{m}^{\mathrm{PC}}$

- A further transformation reduces the RHP for $\mathbf{m}^{\mathrm{PC}}$ to a model RHP whose 2×2 matrix solution $\Phi(\zeta)$ is piecewise analytic in $\mathbb{C}^{\pm}$.
- In each half-plane, the entries of the matrix Φ satisfy ODEs that are obtained from analyticity properties as well as the large-ζ behaviour.

$$\Phi_{11}'' + \left(\frac{\zeta^2}{4} - \beta_{12}\beta_{21} + \frac{i}{2}\right)\Phi_{11} = 0$$

  and similar ODEs for the other Φ_{ij}.
- Additional conditions (conditions at infinity and the jump conditions of Φ) to identify coefficients $\beta_{12}\beta_{21}$.
- The solutions of the ODEs are explicitly calculated in terms of parabolic cylinder functions.

  (Manakov 1974, Its 1982, Deift-Zhou 1993)

Regrouping the transformations, the leading asymptotic behavior of q obtained from the reconstruction formula.

$$q(x,t) \sim t^{-1/2}\alpha(\xi)e^{ix^2/4t - i\nu(\xi)\log(2t)}$$

The parabolic cylinder equation:

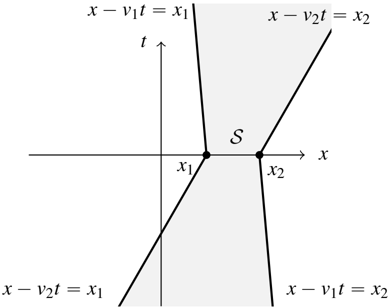
$$y'' + \left(-\frac{z^2}{4} + a + \frac{1}{2}\right) y = 0$$

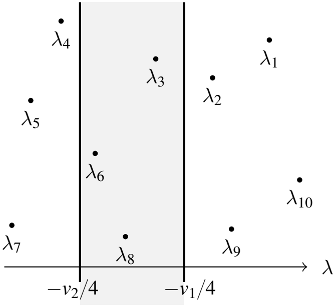
The parabolic cylinder functions $D_a(z)$, $D_a(-z)$, $D_{-a-1}(iz)$, $D_{-a-1}(-iz)$ are entire for any value a. The large-z behavior of $D_a(z)$ is given by

$$D_a(z) \sim \begin{cases} z^a e^{-z^2/4}\,, & |\arg(z)| < \frac{3\pi}{4} \\ z^a e^{-z^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{ia\pi} z^{-a-1} e^{z^2/4}\,, & \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4} \\ z^a e^{-z^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{-ia\pi} z^{-a-1} e^{z^2/4}\,, & -\frac{5\pi}{4} < \arg(z) < -\frac{\pi}{4}. \end{cases}$$

V.2. Focusing NLS: Long-time asymptotics and soliton resolution [Borghese-Jenkins-McLaughlin 2018]

Given $q_0 \in H^{1,1}(\mathbb{R})$. Assume it generates scattering data $\{\rho; \sigma = (\lambda_k, c_k)_{k=1}^N \in \mathbb{C}^{2N}\}$, no spectral singularities.
Fix a space-time cone $x_1 + v_1 t < x < x_2 + v_2 t$. Let $I = [-v_2/4, -v_1/4]$ and $\Lambda(I) = \{\lambda_k,\ \mathrm{Re}\lambda_k \in I\}$, $N_I = |\Lambda(I)|$





As $|t| \to \infty$, inside the space-time cone $\mathcal{S}(v_1, v_2, x_1, x_2)$,

$$q(x,t) \underset{t\to\infty}{\sim} q_{\rm sol}(x,t;\widehat{\sigma}_I) + t^{-1/2} f(x,t) + O(t^{-3/4}).$$

- $q_{\rm sol}(x,t;\widehat{\sigma}_I)$ is a $N(I)$-soliton, with scattering data $\lambda_k \in \Lambda(I)$ and modified connection coefficients $\widehat{c}_k$ due to the soliton-soliton and soliton-radiation interactions.
  In the example of the previous picture, q_{sol} is a 3-soliton.
- $f(x,t)$: dispersive part, given by explicit formula.

VII.2. Asymptotic stability of N-soliton solutions.

Given an N-soliton $q_{\rm sol}(x,t)$ of NLS and parameters $\{\lambda_k^{\rm sol}, c_k^{\rm sol}\}_{k=1}^N$.
Assume initial data $q_0 \in H^{1,1}(\mathbb{R})$ such that

$$\|q_0 - q_{\rm sol}(\,\cdot\,,0)\|_{H^{1,1}(\mathbb{R})} \leqslant \eta_0$$

and with scattering data $\{\rho, \{\lambda_k, c_k\}_{k=1}^N\}$ such that

$$\|\rho\|_{H^{1,1}} + \sum_{k=1}^N |\lambda_k - \lambda_k^{\rm sol}| + |c_k - c_k^{\rm sol}| \leqslant K\eta_1$$

η_0 and η_1 small enough.
As $t \to \infty$, $q(x,t)$ separates into a sum of N 1-solitons

$$\sup_{x\in\mathbb{R}} \left| q(x,t) - \sum_{k=1}^N \mathcal{Q}_{\rm sol}(x,t;\lambda_k,\widetilde{c}_k) \right| \leqslant K|t|^{-1/2},$$

$\mathcal{Q}_{\rm sol}$ are 1-soliton solutions.

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